Kneading Theory

1. Basic definitions.

In this note we consider dynamical systems $([0,1], f)$ where f is *unimodal*; this means

a) $f(0) = f(1) = 0$

b) there is $c = c_f \in [0,1]$ such that f is strictly monotone on $[0,c]$ and on $[c,1]$.

We use symbolic dynamics based on the alphabet $A = \{0, C, 1\}$. We write A^* and A^∞ as usual for the sets of finite and infinite words over A, and we set $A' = A^* \cup A^\infty$. We write ϵ for the empty word, and $|\alpha|$ for the length of the word α. The shift, σ, maps $A' \setminus \{\epsilon\} \to A'$ so that $(\sigma\alpha)_k = \alpha_{k+1}$.

We define an order on A' as follows: For a finite word α we define $\mathrm{slope}(\alpha)$ to be $(-1)^k$ where k is the number of 1's which occur in α. First, define $0 < C < 1$. Then define $a < b$ to mean $a = \alpha x a'$, $b = \alpha y b'$ with $x, y \in A$, satisfying $\mathrm{slope}(\alpha) > 0$ and $x < y$ or $\mathrm{slope}(\alpha) < 0$ and $x > y$. Finally, define $a \leq b$ to mean $a < b$ or $a = b$ or $a \in A^*$ is a prefix of b. Note that, if a or b is in A^*, then $a \leq b$ does not mean "$a < b$ or $a = b$". The following is easy to check:

(1.0) Lemma.

a) If $\alpha, \beta \in A^*$, $a, b \in A'$ then $\alpha < \beta \implies \alpha a < \beta b$.

b) If $\alpha \in A^*$, $b, c \in A'$, and $b < c$ then $\mathrm{slope}(\alpha) > 0 \implies \alpha b < \alpha c$ and $\mathrm{slope}(\alpha) < 0 \implies \alpha b > \alpha c$. The same is true if $<$ and $>$ are replaced by $\leq$ and $\geq$ throughout.

c) $<$ is a linear order on A^∞.

d) On A', $\leq$ is a partial order in which any two words are comparable, satisfying $a = b$ iff $a \leq b$ and $a \geq b$. $\square$

Now we set up the symbolic dynamics. Define $I_0 = [0, c)$, $I_C = \{c\}$, $I_1 = (c, 1]$. Then define I_α, for $\alpha = \alpha_0 \alpha_1 \cdots \in A'$, by

$$I_\alpha = \{\, x \in [0,1] : \forall k < |\alpha| + 1,\ f^k(x) \in I_{\alpha_k} \,\}.$$

Note that, for $a \in A^\infty$, I_a is the nested intersection of I_α where α ranges over the prefixes of a. Also, note that $f I_a \subset I_{\sigma a}$ if $|a| > 0$. We define $S : [0,1] \to A^\infty$ by $S(x) = a$, the *itinerary* of x, iff $x \in I_a$. Note that $f I_\alpha \subset I_{\sigma\alpha}$ for $\alpha \in A' \setminus \{\epsilon\}$; it follows that $S(fx) = \sigma S(x)$. We write $\Sigma_f = S([0,1]) = \{\, a \in A^\infty : I_a \neq \emptyset \,\}$, $L_f = \{\, \alpha \in A^* : I_\alpha \neq \emptyset \,\}$, and $\Sigma'_f = \Sigma_f \cup L_f$.

We say $\alpha \in A'$ is *critical* iff it contains a C, and otherwise we say it is *regular*.

(1.1) Lemma.

a) If $\alpha \in \Sigma'_f$ is critical then I_α is a singleton.

b) If $\alpha \in L_f$ is regular then I_α is an interval which is open in $[0,1]$, and if $n = |\alpha| > 0$ then $f^n | I_\alpha$ is strictly increasing if $\mathrm{slope}(\alpha) > 0$, strictly decreasing if $\mathrm{slope}(\alpha) < 0$.

c) If $\alpha \in \Sigma_f$ then I_α is a non-empty interval (possibly a singleton).

d) If $a, b \in \Sigma'_f$ then $a < b$ if and only if $I_a < I_b$ (i.e., $x < y$ for all $x \in I_a$, $y \in I_b$).

Proof. a) is obvious, and c) is immediate from b). To prove b), proceed by induction: $I_{[illegible]} = (f|I_{[illegible]})^{-1}(I_{[illegible]})$ together with strict monotonicity of $f|I$[illegible] and $f|I_1$ establishes that I_α is an interval, open in $[0,1]$, and the slope statement follows using $f^{n+1}|I_{\alpha_0\ldots\alpha_n} = (f^n|I_{\alpha_1\ldots\alpha_n}) \circ (f|I_{\alpha_0})$.

For d), first note that $I_0 < I_C < I_1$. Suppose $a = \alpha x a'$, $b = \beta y b'$ with $x \neq y$, $x, y \in A$. Since $I_{xa'} \subset I_x$ and $I_{yb'} \subset I_y$ we have $I_{xa'} < I_{yb'}$ iff $x < y$ iff $xa' < yb'$. Suppose $n = |\alpha|$. If slope(α) > 0 then $a < b$ iff $x < y$. On the other hand, f^n is strictly increasing on I_α, which contains both I_a and I_b, so $I_a < I_b$ iff $f^n I_a < f^n I_b$, which holds iff $I_{xa'} < I_{yb'}$, since these are disjoint and contain $f^n I_a$ and $f^n I_b$ respectively. The case of negative slope is handled similarly. Finally, if $a \not< b$ then a is a prefix of b or vice versa, so one of I_a and I_b contains the other. □

(1.2) Lemma.
a) $S(x) < S(y) \implies x < y$,
b) $x \le y \implies S(x) \le S(y)$.

Proof. a) is immediate from (1.1d), and b) follows from a). □

We now define the *kneading sequence* of f to be $K = K_f = S(f(c))$.

(1.3) Lemma.
a) If $a \in \Sigma_f$ *satisfies* $a = \alpha C a'$ *then* $a = \alpha C K$.
b) If $a \in \Sigma_f$ *then* $\sigma^k a \le K$ *for all* $k > 0$.
c) If $K = \eta C K'$ *then* $K = (\eta C)^\infty$.
d) $\sigma^k K \le K$ *for all* $k > 0$.

Proof. c) and d) are just specializations of a) and b), and a) is obvious since $S(c) = CS(f(c)) = CK$. For b), just note that $f(c)$ is the maximum value of f. □

(1.4) Lemma. *If* f *and* g *are unimodal dynamical systems on* $[0,1]$ *which are topologically conjugate then* $K_f = K_g$.

Proof. Suppose $h : [0,1] \to [0,1]$ is a conjugacy from f to g. Then h preserves the set of endpoints, and, since 0 is a fixed point but 1 is not, h maps 0 to 0 and 1 to 1. Therefore h is order preserving. Also, c_f is the only point of $[0,1]$ which does not have a neighborhood on which f is monotone, and a conjugacy must take c_f to the corresponding point c_g for g. Hence h carries the sets I_0, I_C, I_1 for f onto the corresponding sets for g. Since $hf^k(x) = g^k h(x)$ for all x, it is clear that the itinerary of x under f is the same as the itinerary of hx under g. □

We are interested in the converses to these results. That is, given K_f, do (1.3 a,b) characterize Σ_f? Do (1.3 c,d) characterize those sequences which can be kneading sequences? And does the kneading sequence of f determine the topological conjugacy class of f?

2. K_f determines Σ_f.

We shall prove that K_f determines Σ_f under some additional assumptions on f. The troublesome cases are when K_f is periodic, so we do some preliminary analysis in this case.

If α is a finite word ending in 0 or 1 we will use the notation $\hat{\alpha}$ for the word of the same length which differs from α only in the last place, where a 0 is replaced by a 1 or a 1 is replaced by a 0.

Suppose first that K_f is critical. We let η be the maximal regular prefix of $K = K_f$, so $K_f = (\eta C)^\infty$ by (1.3 c), and we set $m = |\eta|$. We introduce the word

τ by the condition that $\tau = \eta 0$ or $\eta 1$, chosen so that $\tau < \eta C$. If $\tau = \eta 0$ then f^m takes I_τ into I_0 and takes its right endpoint fc to c. Hence f^m preserves orientation on I_η, so $\mathrm{slope}(\tau) = \mathrm{slope}(\eta) > 0$. In the alternative case $\tau = \eta 1$ the corresponding argument shows that $\mathrm{slope}(\tau) = -\mathrm{slope}(\eta) > 0$. So

$$\mathrm{slope}(\tau) > 0, \ \mathrm{slope}(\hat\tau) < 0. \tag{2.0}$$

(2.1) Lemma. *Suppose K is critical.*
a) $\sigma^k(\tau^\infty) < K$ and $\sigma^k(\hat\tau\tau^\infty) < K$ for all $k > 0$.
b) K is the only element a of Σ_f satisfying $\tau^\infty < a < \hat\tau\tau^\infty$.

Proof. For $q \ge 0$ define $V_q = I_{\tau^{q+1}} \cup I_K \cup I_{\hat\tau\tau^q}$. We claim that each V_q is a neighborhood of $f(c)$. (This has the consequence that $I_{\tau^{q+1}}$ and $I_{\hat\tau\tau^q}$ are not empty.) This is true for $q = 0$, since $V_0 = I_\tau \cup I_{\eta C} \cup I_{\hat\tau} = I_\eta$ is an open interval (since η is regular) which contains $I_K = \{f(c)\}$. Now we proceed by induction: $f^{m+1} I_K = I_K$, so $W = V_0 \cap f^{-m-1}(V_q) = V_0 \cap f^{-m-1}(V_q \cap \mathrm{Im}\, f)$ is a neighborhood of $f(c)$. Note that $\mathrm{Im}\, f \cap I_{\hat\tau} = \emptyset$, since $f(c)$ is the maximum value of f and $I_K < I_{\hat\tau}$. Hence $V_q \cap \mathrm{Im}\, f = I_{\tau^q} \cup I_K$. Now f^{m+1} is monotone on each of I_τ and $I_{\hat\tau}$, and takes an endpoint $f(c)$ of each of these open intervals onto $f(c)$, so $f^{m+1} I_\tau$ and $f^{m+1} I_{\hat\tau}$ do not contain $f(c)$. Hence $W = (I_\tau \cap f^{-m-1} I_{\tau^{q+1}}) \cup I_K \cup (I_{\hat\tau} \cap f^{-m-1} I_{\tau^{q+1}}) = I_{\tau^{q+2}} \cup I_K \cup I_{\hat\tau\tau^{q+1}} = V_{q+1}$.

Now, for $0 < k \le m+1$, $I_{\sigma^k(\tau^2)}$ and $I_{\sigma^k(\hat\tau\tau)}$ are not empty because they contain $f^k I_{\tau^2}$ and $f^k I_{\hat\tau\tau}$ respectively. Hence $\sigma^k(\tau^2)$ and $\sigma^k(\hat\tau\tau)$ are $\le K$ by (1.1 d); they are $< K$ because they are regular and have length $\ge |\eta C|$. Part a) now follows.

Next, if $\tau^\infty < a = S(x) < \hat\tau\tau^\infty$ then, for some $q \ge 0$, $\tau^{q+1} < a < \hat\tau\tau^q$. Then $I_{\tau^{q+1}} < I_a < I_{\hat\tau\tau^q}$ by (1.1 d), since these are not empty. But, by the claim at the start of the proof, the only point between $I_{\tau^{q+1}}$ and $I_{\hat\tau\tau^q}$ is $f(c)$. $\square$

Remark. We will need at least two more versions of (2.1 b) in the following, all with different hypotheses.

(2.2) Lemma. *Suppose K is critical. If f is C^1 then $V_\infty = I_{\tau^\infty} \cup I_K \cup I_{\hat\tau\tau^\infty}$ is a neighborhood of $f(c)$. In particular, I_{τ^∞} and $I_{\hat\tau\tau^\infty}$ are nontrivial intervals.*

Proof. Recall the neighborhoods V_q introduced in the preceeding proof. Since $K = (\eta C)^\infty$, c is periodic with period $m+1 = |\eta C|$. Then $f(c)$ also has period $m+1$, and $Df^{m+1}(f(c)) = 0$. So $f(c)$ is a periodic sink, and there is a neighborhood U of $f(c)$ in V_0 satisfying $f^{m+1} U \subset U$. But then $U \subset V_0 \cap f^{-m-1} V_0 = V_1$. An induction establishes $U \subset V_q$ for all $q \ge 0$, so $U \subset V_\infty$. $\square$

Now we need some results for regular periodic kneading sequences. For this we introduce the class $\mathcal{D}$ of dynamical systems f defined by

a) f is unimodal;

b) f is C^3 and it has negative Schwarzian derivative: $Sf < 0$;

c) $f'(x) = 0 \iff x = c$;

d) If 0 is a sink then $f^n(c) \to c$ as $n \to \infty$.

The following was proved earlier:

(2.3) Theorem. *If $f \in \mathcal{D}$ has a periodic semi-sink p of period n then there is a non-trivial interval W such that $f^j p \in W$ for some j, $c \in \operatorname{Int} W$, and $f^{nk}x \to f^j p$ as $k \to \infty$ for all $x \in W$. Moreover, for $x \in W$, $f^{2n}x$ is closer to $f^j p$ than x is.*

We also define $\mathcal{D}'$ as the set of $f \in \mathcal{D}$ which satisfy $f''(c) \neq 0$. The following will be proved later:

(2.4) Theorem. *If $f \in \mathcal{D}'$ has no periodic semi-sink then each I_a, for $a \in \Sigma_f$, is a singleton.*

We can now prove

(2.5) Proposition. *Suppose $f \in \mathcal{D}'$. The following are equivalent:*

(1) *f has a periodic semi-sink.*
(2) *K is periodic.*
(3) *K is critical or I_K is a non-trivial interval.*

Proof. (1) $\Longrightarrow$ (2): If c is periodic then so is K, since K is critical iff c is periodic. So we assume that c is not periodic. Let p be a periodic semi-sink of least period n. Let W be the interval guaranteed by (2.3); replacing p by $f^{-j}p$, we have $p \in W$. Let J be the closed subinterval of W with endpoints p and c. We claim that $c \notin f^k J$ for any $k > 0$. Suppose that $k = qn + r > 0$ with $0 \le r < n$, that $f^k J$ contains c, and that k is minimal. Since $f^{mn}x \to f^r p$ as $m \to \infty$ for all $x \in f^k W$ and $f^{mn}c \to p$ we conclude that $r = 0$. Since k is minimal, f^k is monotone on J, and since $f^k p = p$ we have $f^k J \supset J$. Hence $f^k c$ is farther from p than c is, and there is $x \in J$ with $f^k x = c$. Therefore $f^{2k}x = f^k c$ is farther from p than x is, contradicting (2.3). Now it follows that all points in fJ have the same itinerary, so $K = S(fp)$, which is periodic.

(2) $\Longrightarrow$ (3): Suppose K is periodic and I_K is a singleton, so $I_K = \{fc\}$. Then fc is periodic, say of period n. Set $p = f^{n-1}c$. Then $fp = fc$ implies $p = c$ because f takes on its maximum only at c. Hence c is periodic, so K is critical.

(3) $\Longrightarrow$ (1): If K is critical then c is a periodic semi-sink. Suppose K is regular and I_K is a non-trivial interval. Then f^n is monotone on I_K for all $n > 0$ and $f^n I_K \subset I_K$ for some $n > 0$, which implies that f^n has a fixed semi-sink in I_K. $\square$

(2.6) Corollary. *For $f \in \mathcal{D}'$, if I_K is a nontrivial interval then it contains a periodic semi-sink and it contains fc in its interior.*

Proof. The first part of the proof shows that I_K contains the closed interval J between fc and a periodic semi-sink, and that $f^{2n}(fc)$ lies in the interior of J. Since the initerary of fc does not contain c, there is a neighborkood U of fc such that $f^k U$ does not contain c for $0 \le k \le 2n$, with $f^{2n}U \subset J$. Hence $U \subset I_K$, so $fc \in \operatorname{Int} I_K$. $\square$

Now we are ready to prove a partial converse to (1.3 ab).

(2.7) Theorem. *Suppose $f \in \mathcal{D}'$. Then $b \in \Sigma_f$ if and only if*

(1) *If $b = \zeta C b'$ then $b = \zeta C K$.*
(2) *$\sigma^k b \le K$ for all $k > 0$.*
(3) *If K is critical or is regular and not periodic and $\sigma^{n+1} b = K$ for some $n \ge 0$ then $b = \dots C$*

Proof. Necessity of (1) and (2) was shown in (1.3). For (3), suppose $b = S(x)$ and $\sigma^{n+1} b = K$. Set $y = f^n x$ and note that fy and fc are both in I_K. If K is critical or regular and not periodic then I_K is a singleton, by (2.5), so $fy = fc$. Since f takes on its maximum value only at c, we have $y = c$. Hence $b_n = C$.

Now we prove sufficiency. Assume (1) – (3) and suppose $b \notin \Sigma_f$. Define

$$L = \{\, x \in [0,1] : S(x) < b \,\},\ U = \{\, x \in [0,1] : S(x) > b \,\}.$$

We shall show that L and U are open. Assuming this, we finish the proof as follows: We have $L \cup U = [0,1]$ (since $b \notin \Sigma_f$), $L \cap U = \emptyset$, and $0^\infty \in L$, $10^\infty \in U$. This contradicts the connectedness of $[0,1]$.

We give the proof that L is open; the proof for U is entirely similar.

Suppose $x \in L$ and set $a = S(x)$. We shall show $x \in \operatorname{Int} L$. Since $a < b$ we can write $a = \tilde{\alpha} a_n a'$, $b = \tilde{\alpha} b_n b'$ with $\alpha = \tilde{\alpha} a_n < \beta = \tilde{\alpha} b_n$. So $\alpha < b$, and if α is regular we have $x \in I_\alpha \subset L$ with I_α open and we are done.

So assume α is critical. We claim $\tilde{\alpha}$ is regular. Otherwise, suppose $\tilde{\alpha}$ has a prefix of the form γC. Then γC is a prefix of both a and b, so $a = \gamma C K = b$. Therefore $\alpha = \tilde{\alpha} C$ and $a = \alpha K$, and β is regular.

Thus we have $\hat{\beta} < \alpha < \beta$, with $I_{\tilde{\alpha}} = I_{\hat{\beta}} \cup I_\alpha \cup I_\beta$. Since f^{n+1} is monotone on I_β and carries the left endpoint, x, of I_β to the maximum value fc of f, we have $\operatorname{slope}(\beta) < 0$.

Now $\sigma^{n+1} b = b' \le K$. We first suppose $b' < K$. Then there is a prefix δ of K such that $b' < \delta$, so $\beta\delta < \beta b' = b$ since $\operatorname{slope}(\beta) < 0$.

If δ is regular then I_δ is open and non-empty since it contains $fc = f^{n+1}x$, so $W = I_{\tilde{\alpha}} \cap f^{-n-1} I_\delta = I_{\hat{\beta}\delta} \cup I_{\alpha\delta} \cup I_{\beta\delta}$ is a neighborhood of x and $W \subset L$ since $\hat{\beta}\delta < \alpha\delta < \beta\delta < b$. So we are done in this case.

Assume next that δ is critical, so $K = (\eta C)^\infty$. Define τ and V_∞ as in (2.1) and (2.2). Then $f^{m+1} x = fc \in V_\infty$, so $W = I_{\tilde{\alpha}} \cap f^{-m-1} V_\infty$ is a neighborhood of x. Since the image of f misses $I_{\hat{\tau}\tau^\infty}$ we have $W = I_{\hat{\beta}\tau^\infty} \cup I_{\alpha K} \cup I_{\beta\tau^\infty}$. We claim that $b' < \tau^\infty$. Once this is shown we have $\hat{\beta}\tau^\infty < \alpha K < \beta\tau^\infty < b$, so $W \subset L$ as above.

To prove the claim, first suppose $\tau^\infty < b' \le (\eta C)^\infty$. Then let $k \ge 0$ be the largest integer such that τ^k is a prefix of b'. Write $b' = \tau^k b''$. Since $\operatorname{slope}(\tau) > 0$ we have $\tau^\infty < b''$, and since $b'' = \sigma^{(m+1)k} b' \le K$ we have $\tau^\infty < b'' \le (\eta C)^\infty$. Then, since k is minimal and η is a prefix of both τ and ηC, either $\hat{\tau}$ or ηC is a prefix of b''. The former can't happen since $\eta C < \hat{\tau}$, so $b'' = (\eta C)^\infty$ and $b = \beta\tau^k K$. This contradicts (3). Hence the only possibility for $b' \ge \tau^\infty$ is $b' = \tau^\infty$, so $b = \beta\tau^\infty$. Since $I_{\beta\tau^\infty}$ is not empty this contradicts the assumption that $b \notin \Sigma_f$.

Finally we have the case $b' = K$. Then $b = \beta K$ with β regular. So, by (3), K is regular and periodic. By (2.5) and (2.6) I_K contains fc in its interior. Then $W = I_{\tilde{\alpha}} \cap f^{-n-1} I_K = I_{\hat{\beta}K} \cup I_{\alpha K} \cup I_{\beta K}$ is a neighborhood of x. In particular, $I_{\beta K} = I_b$ is non-empty, contradicting $b \notin \Sigma_f$. □

3. Determining kneading sequences.

The following is an intermediate value theorem for kneading sequences. Applied to $f(x, \lambda) = \lambda x(1-x)$ for $\lambda \in [1,4]$ it has the consequence that any sequence satisfying (1.2 a-d) is the kneading sequence of some map.

(3.1) Theorem. *Suppose* $f:[0,1]\times[a,b]\to[0,1]$ *is* C^3, *with* $f_\lambda = f(\cdot,\lambda)\in\mathcal{D}'$ *for all* $\lambda\in[0,1]$, *and suppose all* f_λ *have the same critical point,* c. *Write* K^λ *for the kneading sequence of* f_λ. *If the sequence* M *satisfies*

(1) *If* $M=\eta CM'$ *then* $M=(\eta C)^\infty$,
(2) $\sigma^k M\le M$ *for all* $k>0$,
(3) $K^a\le M\le K^b$,

then $M=K^\lambda$ *for some* $\lambda\in[a,b]$.

Proof. We follow the same outline as in (2.7). So suppose that $M\neq K^\lambda$ for all λ and set

$$L=\{\,\lambda\in[a,b]:K^\lambda<M\,\},\ U=\{\,\lambda\in[a,b]:K^\lambda>M\,\}.$$

We shall show that L and U are open. They are not empty, since $a\in L$ and $b\in U$ by hypothesis; they are disjoint; and their union is $[0,1]$ by assumption. This contradicts connectivity of $[0,1]$.

We give the proof for L; the proof for U is similar.

Suppose $\lambda\in L$, and let ζ be the minimal prefix of K^λ such that $\zeta<M$.

If ζ is regular then there is a neighborhood W of λ such that ζ is a prefix of K^μ for all $\mu\in W$ (since this condition is defined by a fixed finite set of strict inequalities between c and $f_\mu^k(c)$). Then $W\subset L$.

So suppose that ζ is critical. Write $\zeta=\eta\zeta_m$. Since ζ is minimal, η is a prefix of M. Then η is regular, since otherwise M and K^λ share a critical prefix, so $K^\lambda=M$. So $\zeta=\eta C$, so c is periodic with period $m+1$, and $Df_\lambda^{m+1}(c)=0$. We have $K^\lambda=(\eta C)^\infty<M$.

Denoting by I_a^λ the intervals constructed using f_λ, let $\delta>0$ be fixed so that $J=[c-\delta,c+\delta]$ lies in $f_\lambda^{-1}I_\eta^\lambda=I_{0\eta}^\lambda\cup I_{C\eta}^\lambda\cup I_{1\eta}^\lambda$ and $|Df_\lambda^{m+1}(x)|<\frac14$ for all $x\in J$. Let W be a neighborhood of λ such that, for all $\mu\in W$ we have

$$J\subset f_\mu^{-1}I_\eta^\mu,$$
$$|c-f_\mu^{m+1}c|<\frac{\delta}{2},$$
$$|Df_\mu^{m+1}(x)|<\frac12 \text{ for all } x\in J.$$

It follows, by the Banach contraction Theorem (or more elementary means), that, for all $\mu\in W$, f_μ^{m+1} has a unique fixed point p^μ in J, and that $f_\mu^{m+1}(x)$ lies in J and is closer to p^μ than x is for all $x\in J$. Suppose $p^\mu\neq c$. Since $f_\mu J\subset I_\eta^\mu$ and η is regular we have $c\notin f^kJ$ for $0<k\le m$. Hence, since c moves closer to p^μ under f_μ^{m+1} but still lies in J, we continue inductively to the conclusion that c doesn't lie in the image under f_μ^n of the closed interval between c and p^μ, for all $n>0$. It follows that $f_\mu c$ and $f_\mu p^\mu$ have the same itinerary. So, in any case, K^μ is the itinerary of $f_\mu p^\mu$.

Now remember the definition of τ from Section 2. Since $p^\mu\in J$ we have $f_\mu p^\mu\in I_\eta^\mu=I_\tau^\mu\cup I_{\eta C}^\mu\cup I_{\hat\tau}^\mu$. Since $f_\mu p^\mu$ is periodic its itinerary is τ^∞ or $(\eta C)^\infty$ or $\hat\tau^\infty$.

We claim now that $W\subset L$. If no $K^\mu=\hat\tau^\infty$ we have $K^\mu\le(\eta C)^\infty<M$ for all $\mu\in W$, as required. Otherwise, $K^\mu=\hat\tau^\infty$ for some μ, so $M\neq\hat\tau^\infty$. We will show that $(\eta C)^\infty<M<\hat\tau^\infty$ is impossible, so $\hat\tau^\infty<M$, which implies $W\subset L$.

So suppose $(\eta C)^\infty<M<\hat\tau^\infty$. Since η is a prefix of $\hat\tau$ and of ηC it is a prefix of M. So either ηC or $\hat\tau$ is a prefix of M. If ηC is a prefix of M then M

$(\eta C)^\infty = K^\lambda$, a contradiction. So $M = \hat{\tau}M'$. Then $M' = \sigma^{m+1}M \le M < \hat{\tau}^\infty$ and, since slope$(\hat{\tau}) < 0$, we have $M = \hat{\tau}M' > \hat{\tau}^\infty$, a contradiction.

In the proof that U is open we need the similar fact that $\tau^\infty < M < (\eta C)^\infty$ is impossible. We proceed as above to deduce that $M = \tau^k M'$ with $k > 0$ maximal. Then $\tau^\infty < M'$ because slope$(\tau^k) > 0$, and $M' \le M = \tau^k M'$, so τ is a prefix of M', contradicting the choice of k. $\square$

4. Faithful symbolic dynamics.

The main job of this section is the proof of Theorem (2.4). There are various preliminary lemmas. We suppose f is a fixed unimodal dynamical system.

Define $x' = (f|I_1)^{-1}(fx)$ if $x \in I_0$, $x' = (f|I_0)^{-1}(fx)$ if $x \in I_1$, and $c' = c$. Note that $f(x) = f(x')$ and $x \ne x'$ for $x \ne c$. Then $h(x) = x'$ defines a homeomorphism of $[0,1]$ onto itself, with $h^2 = \mathrm{id}$ and $fh = f$. If f is C^1 then the inverse function theorem shows that h is C^1 except, perhaps, at c, with

$$(4.0) \qquad Dh(x) = \frac{Df(x)}{Df(x')} = \frac{1}{Dh(x')}.$$

(4.1) Lemma. *If f is C^2 and $f''(c) \ne 0$ then*

$$0 < \delta = \inf_{x \in I} |Dh(x)| \le 1.$$

Proof. $\delta \le 1$ follows from (4.0).

Let $a = |f''(c)|$ and let U_0 be an open interval containing c such that $\frac{a}{2} < |f''(\xi)| < 2a$ for all $\xi \in U_0$. For $b \in U_0$ close enough to c and less than c we have $b' \in U_0$, so if we define $U = (b, b')$ we have $U \subset U_0$ and $hU = U$. Then $|h'(x)|$ is bounded away from 0 off U, and we only need a bound on U.

By Taylor's theorem we have $|f(x) - f(c)| = \frac{1}{2}|f''(\xi)||x-c|^2$ with ξ between c and x, so $|f(x) - f(c)| \le a|x-c|^2$ for $x \in U$. Applying the same argument to $x' \in U$ we have $|f(x) - f(c)| = |f(x') - f(c)| \ge \frac{a}{4}|x' - c|^2$. Putting these together we get $|x - c| \ge \frac{1}{2}|x' - c|$. Also by the mean value theorem we have $|f'(x)| = |f''(\eta)||x - c|$ with η between c and x. Hence $|f'(x)| \ge \frac{a}{2}|x - c| \ge \frac{a}{4}|x' - c|$ for $x \in U$, and $|f'(x')| \le 2a|x' - c|$. Dividing one by the other yields $|h'(x)| \ge \frac{1}{8}$ for all $x \in U$. $\square$

We use $\ell(J)$ for the length of the interval J. From (4.1) and the mean value theorem we deduce, under the assumptions of (4.1),

(4.2) If J is an interval which does not contain c then $\ell(J') \ge \delta \ell(J)$.

(4.3) Lemma. *If J is an interval, f^n is monotone on J, and $f^n J \subset J$ or J' then f has a periodic semi-sink.*

Proof. If $f^n J \subset J$ then f^n has a fixed semi-sink in $\mathrm{Cl}\, J$ (old result). Otherwise, $f^{n+1}J \subset fJ' = fJ$ and f^n is monotone on fJ so f^n has a fixed semi-sink in $\mathrm{Cl}\, fJ$. $\square$

(4.4) Lemma. *If $f \in \mathcal{D}$ and J is an interval on which f^n is monotone then the minimum value of $|Df^n|$ on* $\mathrm{Cl}\, J$ *occurs at one of the endpoints.*

Proof. This is just a restatement of the basic fact about functions g with negative Schwarzian: Dg has no positive local minimum or negative local maximum in the interior of its domain. $\square$

We shall need the following result, which was proved earlier.

(4.5) Lemma. *Suppose f is C^2, $x \in [0,1]$, and $c \notin \mathrm{Cl}\,\mathcal{O}(x)$. If $x \in J$ and $\Sigma\ell(f^k J)$ is finite then there is a neighborhood V of x such that $\Sigma\ell(f^k V)$ is finite.*

Now we prove (2.4), which we restate:

(2.4) Theorem. *If $f \in \mathcal{D}'$ has no periodic semi-sink then each I_a, for $a \in \Sigma_f$, is a singleton.*

Proof. Suppose $f \in \mathcal{D}'$ does not have a periodic semi-sink and $J = I_a$ is a non-trivial interval. We shall derive a contradiction after a number of claims. Note that a is not critical.

Claim 1. *$f^n J \cap f^m J = \emptyset$ for all $n \neq m$.*

If $f^n J \cap f^{n+k} J \neq \emptyset$ with $k > 0$ then $f^{n+qk} J \cap f^{n+(q+1)k} J \neq \emptyset$ for all $q \geq 0$, so $V = \bigcup_{q=0}^{\infty} f^{n+qk} J$ is an interval, and $f^k V \subset V$. Since f^k has no critical point on each $f^{n+kq} J$, f^k is monotone on V. With (4.3) we have a contradiction.

Claim 2. *$f^n J \cap (f^m J)' = \emptyset$ for all $n \neq m$.*

Otherwise $f^{n+1} J \cap f^{m+1} J \neq \emptyset$.

Claim 3. *If x is an endpoint of J then $c \in \mathrm{Cl}\,\mathcal{O}(x)$.*

From the above, $\Sigma\ell(f^k\,\mathrm{Cl}\,J)$ is finite. If $c \notin \mathrm{Cl}\,\mathcal{O}(x)$ then (4.5) gives a neighborhood V of x such that $\Sigma\ell(f^k V)$ is finite. We can shrink V until this sum is less than the distance from $\mathcal{O}(x)$ to c, so $c \notin f^k V$ for all $k \geq 0$. Hence, since $J \cap V$ is not empty, the points of V have the same itinerary as those in J, so $V \subset J$, a contradiction.

Claim 4. *If x is an endpoint of J then $c \notin \mathcal{O}(x)$.*

Otherwise, if $f^k x = c$ then $\tilde{J} = I_{\sigma^{k+1} a} \supset f^{k+1} J$ is a non-trivial interval with fc as right endpoint. By Claim 3 applied to $\tilde{J}$ there is $n > 0$ such that $f^n \tilde{J}$ meets this interval, contradicting Claim 1 for $\tilde{J}$.

It follows from Claim 4 that the endpoints of J have the same itinerary as points of J, so J is closed.

Notation. If K, L are disjoint closed intervals then we write (K, L) for the open interval of points between K and L, and $[K, L] = K \cup (K, L) \cup L$. If either K or L is a singleton we replace it in this notation with its element. The main point here is that we don't insist that $K < L$.

(4.6) Lemma. *Suppose $d > 0$ and let S be the set of x satisfying*

$$\forall k \text{ s.t. } 0 < k < d,\; f^k x \notin [x, x'] \text{ and } f^d x \in (x, x').$$

If V is a component of S and x is an endpoint of V then $f^d x = x$ or $f^d x = x'$.

Proof. Clearly S is open and an endpoint x of V satisfies $f^k = x$ or $f^k = x'$ for some k, $0 < k \leq d$. Suppose $k < d$ is minimal with $f^k x = x$ and write $d = kq + r$, $0 \leq r < k$. Suppose first that $r = 0$. If $f^k x = x$ we have $f^d x = x$. If $f^k x = x'$ we have $f^{2k} x = f^k(x') = f^{k-1}(f(x')) = f^{k-1}(f(x)) = f^k x = x'$. Continuing by induction leads to $f^d x = x'$. On the other hand, if $r \neq 0$ we have $f^d x = f^r x$ or $f^d x = f^r(x') = f^{r-1} f(x') = f^{r-1} f(x) = f^r x$, and since $r < k$ and k is minimal we have $f^d x = f^r x \notin [x, x']$. So for $y \in V$ close enough to x we have $f^d y \notin [y, y']$, contradicting $y \in S$. $\square$

(4.7) Corollary. $|Df^d(x)| \geq \delta$ *for all* $x \in V$.

Proof. $c \notin V$ because then $f^d c \in (c, c') = \emptyset$, and $c \notin f^k V$ for $0 < k < d$ because if $c = f^k x$ then $c \notin [x, x']$, which is false. So f^d is monotone on V. If $f^d x = x$ for an endpoint x of V then $|Df^d(x)| \geq 1$ since x is not a periodic semi-sink. If $f^d x = x'$ then $f^d(fx) = fx' = fx$, so we have $|Df^d(fx)| \geq 1$. But then

$$\begin{aligned}
|Df^d(x)| =& |Df(x)Df(fx)\dots Df(f^{d-1}x)| \\
=& |Df(fx)\dots Df(f^d x)| \left| \frac{Df(x)}{Df(f^d x)} \right| \\
=& |Df^d(fx)| \left| \frac{Df(x')}{Df(x)} \right| \\
\geq& 1 \cdot |Dh(x')| \geq \delta.
\end{aligned}$$

Now the result follows from (4.4). □

Now we define a sequence k_n by $k_0 = 0$ and $k_n =$ the smallest $l > k_{n-1}$ such that $f^l J \subset (f^{k_{n-1}} J, (f^{k_{n-1}} J)')$. By Claims 1, 2, 3 this is well-defined.

Claim 5. $\ell(f^{k_n} J) \geq \delta \ell(f^{k_{n-1}} J)$.

This follows from (4.7) with $d = k_n - k_{n-1}$ upon noting that $f^{k_{n-1}} J \subset S$.

Now for $m > 0$ let α be the prefix of a of length m. Since I_a is the nested intersection of the I_α's we can find m_0 such that, for all $m \geq m_0$, $\ell(I_\alpha) < (1 + \delta^2)\ell(I_a)$. We set $m = k_n$ and assume n is large enough that $m \geq m_0$. The goal is to show that $\ell(f^{k_n} J) \geq \ell(f^{k_{n-1}} J)$. This contradicts Claim 1 and finishes the proof.

For this m write I_α as the disjoint union of the nontrivial intervals L, J, and R, with $L < J < R$, and write $\xi < \eta$ for the endpoints of I_α.

From the choice of m_0 we have

Claim 6. $\dfrac{\ell(L)}{\ell(J)}, \dfrac{\ell(R)}{\ell(J)} < \delta^2$.

Claim 7. $\dfrac{\ell(f^m J)}{\ell(J)} \geq$ *either* $\dfrac{\ell(f^m L)}{\ell(L)}$ *or* $\dfrac{\ell(f^m R)}{\ell(R)}$.

To prove this, suppose $|Df^m|$ takes on its minimum on J at the left endpoint (using (4.4)). Then the maximum of $|Df^m|$ on L must be equal to this minimum value, since otherwise $|Df^m|$ would have a non-zero local minimum in $L \cup J$. Now apply the mean value theorem.

Claim 8. $c \in f^m \operatorname{Cl} I_\alpha = f^m[\xi, \eta]$.

Assume not. Then either $f^m \operatorname{Cl} L$ or $f^m \operatorname{Cl} R$ is in the interval $(f^m J, c)$. Assume the latter. Then for some $j < m$ we have $f^j \eta = c$. Since $j < m$, $f^j J$ lies in the complement of $[f^m J, (f^m J)']$. Hence $f^j R$ or $(f^j R)'$ contains $f^m J$ and we apply (4.3) to f^{m-j} on $f^j R$ to find a periodic semi-sink.

Henceforth we assume

$$c \in f^m(\operatorname{Cl} R). \tag{4.8}$$

Claim 9. *$f^m L \supset f^{k_{n-1}} J$ or $(f^{k_{n-1}} J)'$.*

Write $K = f^{k_{n-1}} J$, $M = (K, c)$. For some $i \leq m$ we have $f^i \xi = c$ and (4.8) implies $i < m$. Then $f^i J$ is in the complement of $[K, K']$ unless $i = k_{n-1}$, so $f^i L$ contains $M \cup K$ or $M' \cup K'$ unless $i = k_{n-1}$. On the other hand, $f^m J \subset M$ or M', and $f^m L$ is on the same side of $f^m J$ as K or K' respectively. We now assume that $f^m L$ does not contain K or K'; by the preceeding discussion this implies that $f^m L$ lies in $M \cup K$ or $M' \cup K'$. If $i \neq k_{n-1}$ this leads to a periodic semi-sink via (4.3) applied to f^{m-i} on $f^i L$. If $i = k_{n-1}$ then $f^i L = M$ and $f^{m-i}(M \cup K) = f^m L \cup f^m J \subset M \cup K$ or $M' \cup K'$, and we apply (4.3) to f^{m-i} on $M \cup K$ to produce a periodic semi-sink.

Claim 10. $\dfrac{\ell(f^m R)}{\ell(R)} > \dfrac{\ell(f^m J)}{\ell(J)} \geq \dfrac{\ell(f^m L)}{\ell(L)}$.

By Claim 8, $f^m R$ contains $f^{k_{m+1}} J$ or $(f^{k_{m+1}} J)'$. From Claim 5 we have $\ell(f^{k_{m+1}} J) \geq \delta \ell(f^m J)$ and from (4.2) we have $\ell((f^{k_{m+1}} J)') \geq \delta \ell(f^{k_{m+1}} J)$. Considering both cases, we have $\ell(f^m R) \geq \delta^2 \ell(f^m J)$. From Claim 6, $\ell(R) < \delta^2 \ell(J)$, so division yields the first inequality of the claim. The second now follows from Claim 7.

We can now finish the proof by showing

Claim 11. $\ell(f^{k_n} J) \geq \ell(f^{k_{n-1}})$.

Let $r = \ell(f^{k_{n-1}})$. Then

$$
\begin{aligned}
\delta r &\leq \ell(f^m L) && \text{by Claim 9 and (4.2)} \\
&\leq \frac{\ell(L)}{\ell(J)} \ell(f^m J) && \text{by Claim 10} \\
&\leq \delta^2 \ell(f^m J) && \text{by Claim 6.}
\end{aligned}
$$

Hence $\ell(f^m J) \geq \delta^{-1} r > r$, as required. $\square$