SUSLIN SETS

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1. Introduction

This is just an account of the basic facts about Suslin sets, which were invented (but not named) by Suslin in [3]. Suslin lived 1894–1919, and only published this one paper (which I have not read). Most of this material was found in [1] except for the measure theory in Section 7, which is from [2]. These books have references to the original papers.

2. Preliminaries

We use F_x or $F(x)$ interchangeably for the value of a function F at x. We write Y^X for the set of all functions from X to Y.

2.1. *Suppose that $S\colon X \times Y \to \mathcal{P}(Z)$ is a function. Then*

$$\bigcap_{x\in X} \bigcup_{y\in Y} S_{xy} = \bigcup_{f\in Y^X} \bigcap_{x\in X} S_{xf_x}$$

Proof. Suppose z is in the RHS. Then, for some $f \in Y^X$, $z \in S_{xf_x}$ for all $x \in X$. Hence for each $x \in X$ there is $y = f_x$ so that $z \in S_{xy}$, so z is in the LHS.

Suppose z is in the LHS. Then for each $x \in X$ there is some $y \in Y$ so that $z \in S_{xy}$. Hence for each $x \in X$ the set $T_x = \{\, y : z \in S_{xy} \,\}$ is not empty. By AC there is a function $f \in Y^x$ so that, for all $x \in X$, $f_x \in T_x$. Hence, for all $x \in X$, $z \in S_{xf_x}$, so z is in the RHS. □

We write $\mathbb{P}$ for the set of positive integers. On $\mathbb{P}$ we define a binary operator $*$ by $k * j = 2^{k-1}(2j - 1)$. Since every positive integer can be factored uniquely as a product of a power of 2 and an odd integer we have:

2.2. *$(k, j) \mapsto k * j$ is a bijection of $\mathbb{P}^2$ onto $\mathbb{P}$.*

By a *word* of length n over $\mathbb{P}$ we mean an element w of $\mathbb{P}^n$, written as $w = \langle w_1, w_2, \ldots, w_n\rangle$. The set of all words of positive length is written $\mathbb{P}^+$.

2.3. *$s = \langle s_1, s_2, \ldots, s_k\rangle \mapsto k * s_1 * s_2 * \cdots * s_k$ defines a bijection of $\mathbb{P}^+$ onto $\mathbb{P}$ (where $*$ is considered to be right associative).*

3. Index schemes

By an *index set* I we mean any non-empty countable set. An *index scheme* over an index set I is a collection α of non-empty subsets of I. If α is an index scheme over I and C is a set-valued function defined on I we define

$$\Phi_\alpha(C) = \bigcup_{A\in\alpha} \bigcap_{i\in A} C_i.$$

If $\mathcal{C}$ is a collection of sets then we define $\mathcal{C}_\alpha = \{\, \Phi_\alpha(C) : C \in \mathcal{C}^I \,\}$.

Here are three special index schemes. First, with index set $\mathbb{P}$ let $\delta = \mathbb{P}$ and let $\sigma = \{\, \{k\} : k \in \mathbb{P} \,\}$. Then $\mathcal{C}_\delta$ is the set of all countable intersections of elements of $\mathcal{C}$ and $\mathcal{C}_\sigma$ is the set of all countable unions of elements of $\mathcal{C}$, so this notation is consistent with earlier usage. The other special scheme uses the index set $\mathbb{P}^+$. If $a \in \mathbb{P}^\mathbb{P}$ is any sequence we define Z_a to be the set of all initial words of a. That is, $Z_a = \{\, \langle a_1, a_2, \ldots, a_n \rangle : n \in \mathbb{P} \,\}$. We define $\zeta = \{\, Z_a : a \in \mathbb{P}^\mathbb{P} \,\}$; this is the *Suslin* scheme.

Suppose α is an index scheme with index set I, J is another index set, and $h\colon I \to J$. For $A \subset I$ define $hA = \{\, h(i) : i \in A \,\}$ and define $h\alpha = \{\, hA : A \in \alpha \,\}$. This is an index scheme with index set J.

3.1. *$\mathcal{C}_{h\alpha} \subset \mathcal{C}_\alpha$, with equality if h is a bijection.*

Proof. Given $C \in \mathcal{C}^J$ define $h^*C \in \mathcal{C}^I$ by $(h^*C)_i = C_{h(i)}$. Now

$$\Phi_{h\alpha}(C) = \bigcup_{B \in h\alpha} \bigcap_{j \in B} C_j = \bigcup_{A \in \alpha} \bigcap_{j \in hA} C_j = \bigcup_{A \in \alpha} \bigcap_{i \in A} C_{h(i)} = \Phi_\alpha(h^*C).$$

Then the inclusion $\mathcal{C}_{h\alpha} \subset \mathcal{C}_\alpha$ is immediate from $\Phi_{h\alpha}(C) = \Phi_\alpha(h^*C)$. □

3.2. *If $\alpha \neq \emptyset$ then $\mathcal{C} \subset \mathcal{C}_\alpha$.*

Proof. Let $\varepsilon = \{1\}$. This is an index scheme with index set $\{1\}$, and $\mathcal{C} = \mathcal{C}_\varepsilon$ for all $\mathcal{C}$. Given any other index scheme with index set I, let h map I to $\{1\}$. Then $h\alpha = \varepsilon$. □

3.3. *$\mathcal{C}_\delta \subset \mathcal{C}_\zeta$ and $\mathcal{C}_\sigma \subset \mathcal{C}_\zeta$*

Proof. For the first define $h\colon \mathbb{P}^+ \to \mathbb{P}$ by $\langle w_1, w_2, \ldots, w_n \rangle \mapsto w_n$, and for the second use $\langle w_1, w_2, \ldots, w_n \rangle \mapsto w_1$. □

Suppose α and β are index schemes, with index sets I and J respectively. Let $K = J \times I$. If $B \in \beta$ and $F \in \alpha^B$ define $G(B, F) = \{\, (j, i) \in K : j \in B,\ i \in F_j \,\}$. Define $\alpha\beta = \{\, G(B, F) : B \in \beta,\ F \in \alpha^B \,\}$. This is an index scheme with index set K.

3.4. *$\mathcal{C}_{\alpha\beta} = (\mathcal{C}_\alpha)_\beta$.*

Proof. Suppose E is in the RHS. Then $E = \Phi_\beta(C)$ where $C \in (C_\alpha)^J$. Hence for each $j \in J$ there is $D_j \in \mathcal{C}^I$ so that $C_j = \Phi_\alpha(D_j)$. We consider $D \in \mathcal{C}^{J \times I} = \mathcal{C}^K$. Using 2.1,

$$\begin{aligned} E =& \Phi_\beta(C) = \bigcup_{B \in \beta} \bigcap_{j \in B} C_j = \bigcup_{B \in \beta} \bigcap_{j \in B} \Phi_\alpha(D_j) = \bigcup_{B \in \beta} \bigcap_{j \in B} \bigcup_{A \in \alpha} \bigcap_{i \in A} D_{ji} \\ =& \bigcup_{B \in \beta} \bigcup_{F \in \alpha^B} \bigcap_{j \in B} \bigcap_{i \in F_j} D_{ji} = \bigcup_{G \in \alpha\beta} \bigcap_{(j,i) \in G} D_{ji} = \Phi_{\alpha\beta}(D). \end{aligned}$$

Thus E is in the LHS.

Conversely, if E is in the LHS then $E = \Phi_{\alpha\beta}(D)$ for some $D \in \mathcal{C}^K = \mathcal{C}^{J \times I}$. We define, for each $j \in J$, $D_j \in \mathcal{C}^I$ by $D_j(i) = D_{ji}$. Further, we define $C \in (C_\alpha)^J$ by $C_j = \Phi_\alpha(D_j)$. Then the calculation above can be read backwards, showing that E is in the RHS. □

Theorem 3.5. $\mathcal{C}_{\zeta\zeta} = \mathcal{C}_\zeta$.

Proof. Using 3.2 we only have to show $\mathcal{C}_{\zeta\zeta} \subset \mathcal{C}_\zeta$, and for this we will use 3.1. The index set for ζ is $\mathbb{P}^+$ and the index set for $\zeta\zeta$ is $\mathbb{P}^+ \times \mathbb{P}^+$, so we need a map $h\colon \mathbb{P}^+ \to \mathbb{P}^+ \times \mathbb{P}^+$. Let $w = \langle w_1, w_2, \ldots, w_n \rangle$ be in $\mathbb{P}^+$. Write $n = k * j$, and write $w_p = y_p * x_p$ for $1 \le p \le n$. We define $h(w) = (v, u)$ where v has length k and is defined by $v_m = y_{m*1}$ and u has length j and is defined by $u_\ell = x_{k*\ell}$. We need to show that $h\zeta = \zeta\zeta$.

First, suppose that $c \in \mathbb{P}^{\mathbb{P}}$, defining $Z_c = \{\, \langle c_1, \ldots, c_n \rangle : n \in \mathbb{P} \,\}$ in ζ. We factor $c_p = y_p * x_p$ as above. Then $b_m = y_{m*1}$ defines a sequence b, so an element $B = Z_b$ of ζ. To each element $s = \langle b_1, \ldots, b_k \rangle$ of B we associate the sequence a_s defined by $a_{s\ell} = x_{k*\ell}$. Using a_s we define $F \in \zeta^B$ by $F(s) = Z_{a_s}$. Now consider hZ_c. This is the set of all pairs $(s, \langle a_{s1}, a_{s2}, \ldots, a_{sj} \rangle)$ where $s = \langle b_1, \ldots, b_k \rangle$. In other words, hZ_c consists of all pairs (s,t) where $s \in B$ and $t \in F(s)$. Hence $hZ_c \in \zeta\zeta$.

Now consider W in ζ. Then there is $B \in \zeta$ and there is a function $F \in \zeta^B$ so that $W = \{\, (s,t) : s \in B,\, t \in F(s) \,\}$. Since $B \in \zeta$ there is a sequence b so that $B = Z_b = \{\, \langle b_1, \ldots, b_k \rangle : k \in \mathbb{P} \,\}$. Furthermore, for each $s = \langle b_1, \ldots, b_k \rangle$ in B the set $F(s)$ is in ζ so it has the form Z_{a_s} for some sequence a_s. We define the sequence $c \in \mathbb{P}^{\mathbb{P}}$ by the rule $c_{k*j} = b_k * a_{sj}$, where $s = \langle b_1, \ldots, b_k \rangle$. It should now be clear that $hZ_c = W$, so $W \in h\zeta$.

From these two arguments we have $h\zeta \subset \zeta\zeta$ and $\zeta\zeta \subset h\zeta$, finishing the proof. □

We need one more operation, a slight generalization of $\alpha \mapsto \alpha\sigma$. Suppose J is an index set and, for each $j \in J$, α_j is an index scheme with index set I_j. Define $K = \bigcup_{j\in J} \{j\} \times I_j$ and define $\alpha\Sigma$ to consist of all sets $\{j\} \times A$ where $A \in \alpha_j$. Then K is countable and $\alpha\Sigma$ is an index scheme with index set K.

3.6. *$E \in \mathcal{C}_{\alpha\Sigma}$ iff $E = \bigcup_{j\in J} E_j$ where each E_j is in $\mathcal{C}_{\alpha_j}$.*

Proof. $E \in \mathcal{C}_{\alpha\Sigma}$ iff $E = \Phi_{\alpha\Sigma}(D)$ for some $D \in \mathcal{C}^K$. Then, writing $W = \{j\} \times A$,

$$E = \bigcup_{W \in \alpha\Sigma} \bigcap_{(j,i)\in W} D_{ji} = \bigcup_{j\in J} \bigcup_{A\in\alpha_j} \bigcup_{i\in A} D_{ji} = \bigcup_{j\in J} E_j,$$

with $E_j = \bigcup_{A\in\alpha_j} \bigcup_{i\in A} D_{ji} \in \mathcal{C}_{\alpha_j}$. The reverse implication follows similarly. □

4. Borel and Suslin sets

Suppose X is a metric space. Then $\mathcal{G}$ and $\mathcal{F}$ are the collections of open, respectively closed subsets of X. When necessary we indicate, in this and other collections of sets, the dependence on X by notation like $\mathcal{G}(X)$.

For $x \in X$ and $r > 0$ we write $B(x,r)$ for $\{\, y \in X : d(y,x) < r \,\}$, and for $E \subset X$ and $r > 0$ we write $B(E,r)$ for the union of all $B(x,r)$ for $x \in E$.

4.1. $\mathcal{F} \subset \mathcal{G}_\delta$.

Proof. The reader may check that if F is closed then $F = \bigcap_n B(F, \frac{1}{n})$. □

A σ-algebra in $\mathcal{P}(X)$ is a subset $\mathcal{A}$ of $\mathcal{P}(X)$ which contains $\emptyset$ and is closed under complementation with respect to X and under countable unions (that is, $\mathcal{A}_\sigma \subset \mathcal{A}$). From de Morgan's laws it also follows that $\mathcal{A}_\delta \subset \mathcal{A}$. The intersection of all σ-algebras containing a collection $\mathcal{C} \subset \mathcal{P}(X)$ is a σ-algebra, known as the σ-algebra *generated* by $\mathcal{C}$. We write $\mathcal{C}^*$ for the σ-algebra generated by $\mathcal{C}$.

The *Borel* class $\mathcal{B}$ is the σ-algebra generated by $\mathcal{G}$.

4.2. *$\mathcal{B}$ is the smallest collection in $\mathcal{P}(X)$ containing $\mathcal{G}$ and closed under countable intersection and countable union.*

Proof. $\mathcal{B}$ is closed under countable union and countable intersection. Suppose $\mathcal{C} \subset \mathcal{B}$ is closed under countable union and countable intersection and contains $\mathcal{G}$. Define $\mathcal{C}_0 = \{ E \in \mathcal{C} : X \setminus E \in \mathcal{C} \}$. Then $\mathcal{C}_0$ contains $\emptyset$ and is closed under complementation, and closure under countable unions follows easily using de Morgan. Thus $\mathcal{C}_0$ is a σ-algebra in $\mathcal{C}$. Moreover, it contains $\mathcal{G}$ since if $E \in \mathcal{G}$ then $E \in \mathcal{C}$ and $X \setminus E \in \mathcal{F} \subset \mathcal{G}_\delta \subset \mathcal{C}_\delta \subset \mathcal{C}$ (using 4.1). Hence $\mathcal{C}_0 = \mathcal{B}$ and therefore $\mathcal{C} = \mathcal{B}$. □

We define the *Suslin* class $\mathcal{Z}$ as $\mathcal{G}_\zeta$. (Suslin sets are also known as *analytic* sets.) By 3.3 and 3.5 we have $\mathcal{Z}$ closed under countable union and intersection, so

4.3. *$\mathcal{B} \subset \mathcal{Z}$.*

The same kinds of arguments as the last two lead to

4.4. *Suppose $\mathcal{C} \subset \mathcal{B}$ and $\mathcal{C}_\sigma \supset \mathcal{G}$. Then*

(1) *$\mathcal{B} = \mathcal{C}^*$.*
(2) *$\mathcal{B}$ is the smallest collection containing $\mathcal{C}$ and closed under countable union and intersection.*
(3) *$\mathcal{Z} = \mathcal{C}_\zeta$.*

Suppose that Y is a metric space and X is a subspace. In order to talk about relative Borel and Suslin sets we introduce (temporarily) the following terminology. If $\mathcal{C}$ is a collection of subsets of Y we define $\mathcal{C}' = \{ E \cap X : E \in \mathcal{C} \}$ and $\mathcal{C}'' = \{ E \in \mathcal{C} : E \subset X \}$.

4.5. *$\mathcal{C}'' \subset \mathcal{C}'$, and if α is an index scheme with index set I then $(\mathcal{C}_\alpha)' = (\mathcal{C}')_\alpha$.*

Proof. If $E \in \mathcal{C}''$ then $E \in \mathcal{C}$ and $E = E \cap X$ so $E \in C'$.

If $E \in (\mathcal{C}_\alpha)'$ then $E = \Phi_\alpha(C) \cap X = \Phi_\alpha(D)$ where $C \in \mathcal{C}^I$ and D is defined by $D_i = C_i \cap X$. Conversely, if $E \in (\mathcal{C}')_\alpha$ then $E = \Phi_\alpha(D) = \Phi_\alpha(C) \cap X$ where $D \in (\mathcal{C}')^I$ and $C \in \mathcal{C}^I$ is determined by the Axiom of Choice so that $D_i = C_i \cap X$ for all i. □

4.6. *Suppose X is a Borel set in Y.*

(1) *$\mathcal{B}(X) = \mathcal{B}(Y)' = \mathcal{B}(Y)''$.*
(2) *$\mathcal{Z}(X) = \mathcal{Z}(Y)' = \mathcal{Z}(Y)''$.*

Proof. Since $\mathcal{B}(Y)$ and $\mathcal{Z}(Y)$ are closed under intersection with Borel sets we have $\mathcal{B}(Y)' \subset \mathcal{B}(Y)''$ and $\mathcal{Z}(Y)' \subset \mathcal{Z}(Y)''$, so, using 4.5, $\mathcal{B}(Y)' = \mathcal{B}(Y)''$ and $\mathcal{Z}(Y)' = \mathcal{Z}(Y)''$.

Next, $\mathcal{B}(Y)'$ is easily seen to be a σ-algebra in $\mathcal{P}(X)$ and it contains $\mathcal{G}(X) = \{ E \cap X : E \in \mathcal{G}(Y) \}$, so $\mathcal{B}(Y)' \supset \mathcal{B}(X)$. Hence $\mathcal{B}(X) \subset \mathcal{B}(Y)'' \subset \mathcal{B}(Y)$ and, for the same reasons, $\mathcal{B}(Y \setminus X) \subset \mathcal{B}(Y)$. Let $\mathcal{A}$ be the collection of all $E \subset Y$ so that $E \cap X \in \mathcal{B}(X)$ and $E \setminus X \in \mathcal{B}(Y \setminus X)$. It is easy to check that $\mathcal{A}$ is a σ-algebra. Suppose G is open in Y. Then $G \cap X \in \mathcal{G}(X) \subset \mathcal{B}(X)$ and similarly $G \setminus X = G \cap (Y \setminus X) \in \mathcal{B}(Y \setminus X)$, so G is in $\mathcal{A}$. Thus $\mathcal{A}$ is a σ-algebra in $\mathcal{P}(Y)$ containing $\mathcal{G}(Y)$ so $\mathcal{A}$ contains $\mathcal{B}(Y)$. Now if $E \in \mathcal{B}(Y)'' \subset \mathcal{B}(Y)$ then $E \in \mathcal{A}$ so $E = E \cap X \in \mathcal{B}(X)$. This completes the proof that $\mathcal{B}(X) = \mathcal{B}(Y)''$.

Finally, $\mathcal{Z}(X) = (\mathcal{B}(X))_\zeta = (\mathcal{B}(Y)')_\zeta$, which equals $(\mathcal{B}(Y)_\zeta)' = \mathcal{Z}(Y)'$ by 4.5. □

5. Non-Borel and Non-Suslin Sets

If $J = [a, b]$ is an interval with $a < b$ of length $\lambda = b - a$ and $s \in \mathbb{P}$ we define $J_s = \left[b - \frac{\lambda}{3^{s-1}}, b - \frac{2\lambda}{3^s}\right]$. This is a closed subinterval of J of length $\lambda/3^s$ and the collection of such intervals is pairwise disjoint; in fact, $J_s < J_t$ if $s < t$.

We start with $J = [0, 1]$ and define J_s, for $s \in \mathbb{P}^k$, recursively by $J_{st} = (J_s)_t$ for $s \in \mathbb{P}^{k-1}$ and $t \in \mathbb{P}$. Note that, for s and t in $\mathbb{P}^+$, J_s and J_t are disjoint unless one of s, t is a prefix of the other. If s is a prefix of t then $J_s \supset J_t$; if neither is a prefix of the other then $s < t$ or $t < s$ in the lexicographic order, and $J_s < J_t$ if $s < t$.

We let $X_k = \bigcup_{s\in\mathbb{P}^k} J_s$ and $X = \bigcap_{k\in\mathbb{P}} X_k$. Each X_k is in $\mathcal{F}_\sigma$ so X is in $\mathcal{F}_{\sigma\delta} \subset \mathcal{B}$. Corresponding to $s \in \mathbb{P}^\mathbb{P}$ there is a nested sequence of closed intervals $\langle J_{s_1 s_2 \ldots s_k} : k \in \mathbb{P}\rangle$ with length approaching 0, so there is a unique point $\psi(s)$ in the intersection. It is easily verified that ψ is a bijection of $\mathbb{P}^\mathbb{P}$ onto X. We will abuse this bijection by writing $x = \langle x_1, x_2, \ldots\rangle$ where $x_k = s_k$ and $s = \psi^{-1}(x)$.

Now suppose α is any index scheme with index set I and choose an injection $h : I \to \mathbb{P}$. Also suppose $\mathcal{C}$ is a countable collection of sets, enumerated as $\{ D_n : n \in \mathbb{P} \}$. If $x = \langle x_1, x_2, \ldots\rangle \in X$ we define $\Phi_\alpha(x) = \Phi_\alpha(C)$ where $C \in \mathcal{C}^I$ is defined by $C_i = D_{x_{h(i)}}$. Thus $\Phi_\alpha(x) \in \mathcal{C}_\alpha$. Moreover, since any map $I \to \mathcal{C}$ can be represented in the form above,

5.1. Φ_α *is a surjection of* X *onto* $\mathcal{C}_\alpha$.

For $s \in \mathbb{P}^+$ define $B_s = J_s \cap X$ and let $\mathcal{C} = \{ B_s : s \in \mathbb{P}^+ \}$. These sets are closed in X, and $\mathcal{C}_\sigma \supset \mathcal{G}(X)$. To see this just note that if G is open in $\mathbb{R}$ and $x \in X \cap G$ then there is J_s containing x and of diameter less than the distance from x to $\mathbb{R}\setminus G$ so $x \in B_s \subset G \cap X$. We use the bijection given by 2.3 between $\mathbb{P}^+$ and $\mathbb{P}$, and define $D_n = B_s$ where $s = \langle s_1, s_2, \ldots, s_k\rangle$ and $n = k * s_1 * s * 2 * \cdots * s_k$. [Actually, the exact form of the bijection is not important.]

Theorem 5.2. *Define* $Q_\alpha = \{ x \in X : x \in \Phi_\alpha(x) \}$ *and* $R_\alpha = X \setminus Q_\alpha = \{ x \in X : x \notin \Phi_\alpha(x) \}$. *Then* $Q_\alpha \in \mathcal{C}_{\sigma\alpha}$ *and* $R_\alpha \notin \mathcal{C}_\alpha$.

Proof. This of course is just a variation on the classical Cantor construction of an uncountable set.

If R_α were in $\mathcal{C}_\alpha$ then it would equal $\Phi_\alpha(a)$ for some $a \in X$. But then $a \in R_\alpha$ if and only if $a \notin R_\alpha$ by the definition of R_α, a contradiction.

Now for each j and n in $\mathbb{P}$ define $E_{jn} = \{ x \in X : x_j = n \}$. This is just the union of all sets B_{sn} where $s \in \mathbb{P}^{j-1}$ so this is in $\mathcal{C}_\sigma$. We define $F_j = \bigcup_{n\in\mathbb{P}} E_{jn} \cap D_n$. Then each $B_{sn} \cap D_n$ is a union of various B_t, so F_j is also in $\mathcal{C}_\sigma$. Note that $x \in F_j$ if and only if there is n so that $x_j = n$ and $x \in D_n$, so $F_j = \{ x \in X : x \in D_{x_j} \}$. We claim that $Q_\alpha = \Phi_\alpha(F')$ where $F'_i = F_{h(i)}$. Since each F_j is in $\mathcal{C}_\sigma$ this will finish the proof that $Q_\alpha \in \mathcal{C}_{\sigma\alpha}$.

For the claim, first suppose $x \in Q_\alpha$. Then $x \in \Phi_\alpha(x) = \bigcup_{A\in\alpha} \bigcap_{i\in A} D_{x_{h(i)}}$. Hence, for some $A \in \alpha$, x is in $D_{x_{h(i)}}$ for each $i \in A$. But $x \in D_{x_{h(i)}}$ is equivalent to $x \in F_{h(i)} = F'_i$, so we have $x \in F'_i$ for all $i \in A$, so $x \in \Phi_\alpha(F')$. The steps are reversible to prove the opposite inclusion. □

Corollary 5.3. Q_ζ *is a Suslin set in* $\mathbb{R}$ *but not a Borel set;* R_ζ *is not a Suslin set in* $\mathbb{R}$.

Proof. Note that, since X is a Borel set, there is no difference between stating this in $\mathbb{R}$ or in X. The only part not immediate from the theorem is that Q_ζ is not a Borel set; but if it were then its complement, R_ζ, would be a Borel set and hence a Suslin set. □

Note that X has lebesgue measure 0, so Q_ζ is an example of a Lebesgue measurable set which is not a Borel set. But in fact this argument is not needed; see Section 7.

6. Cardinals and Ordinals

We use $\mathfrak{d}$ for the cardinal number of the integers, and $\mathfrak{c} = 2^{\mathfrak{d}}$ for the cardinal number of the reals.

6.1. *The Borel and Suslin classes in $\mathbb{R}$ have cardinality $\mathfrak{c}$.*

Proof. The cardinality of $\mathcal{B}$ is at least $\mathfrak{c}$ since it contains $\mathcal{P}(\mathbb{P})$. On the other hand, 5.1 applied to ζ and the countable collection of all intervals in $\mathbb{R}$ with rational endpoints shows that the cardinality of $\mathcal{Z}$ is at most that of $X \subset \mathbb{R}$. □

Since $\mathcal{P}(\mathbb{R})$ has cardinality $2^{\mathfrak{c}}$, "most" subsets of $\mathbb{R}$ are not Suslin sets. Moreover, the class $\mathcal{M}$ of Lebesgue measurable sets in $\mathbb{R}$ has cardinality $2^{\mathfrak{c}}$ since it contains all subsets of the standard Cantor set. Thus "most" Lebesgue measurable sets in $\mathbb{R}$ are not Suslin sets.

Now consider ordinal numbers. We identify the ordinal x with the set of all ordinals preceding x. Thus the first infinite ordinal ω is the set of finite ordinals, and the first uncountable ordinal Ω is the set of all countable ordinals. We write the successor of the ordinal x as $s(x)$.

We define a sequence of index schemes $\xi(x)$, indexed by the countable ordinals, as follows. We start with $\xi(0) = \{1\}$ with index set $\{1\}$, so $\mathcal{C}_{\xi(0)} = \mathcal{C}$ for all $\mathcal{C}$. Proceding by transfinite recursion, we define $\xi(s(x)) = \xi(x)\delta\sigma$. If x is a countable limit ordinal then x is a countable set and $\xi(y)$ is an index scheme for all $y \in x$ so we can define $\xi(x) = \xi\Sigma$ using the notation of 3.6. (More precisely, $\xi(x) = (\xi\,|_x)\Sigma$.) The following is a simpler description:

6.2. *Suppose $\emptyset \in \mathcal{C}$. If x is a countable limit ordinal then $\mathcal{C}_{\xi(x)} = (\mathcal{C}^0_{\xi(y)})_\sigma$ where $\mathcal{C}^0_{\xi(y)} = \bigcup_{y<x} \mathcal{C}_{\xi(y)}$.*

Proof. According to 3.6, $\mathcal{C}_{\xi(x)}$ consists of all unions $E = \bigcup_{y \in x} E_y$ where $E_y \in \mathcal{C}_{\xi(y)}$. Since x is countable this is a countable union, and since the sets $\mathcal{C}_{\xi(y)}$ are nested each E_y is in $\mathcal{C}^0_{\xi(y)}$, proving $\mathcal{C}_{\xi(x)} \subset (\mathcal{C}^0_{\xi(y)})_\sigma$.

Now suppose $E = \bigcup_{k \in \mathbb{P}} E_k$ with each $E_k \in \mathcal{C}^0_{\xi(y)}$. We define $y(k)$ as the minimum y such that $E_k \in \mathcal{C}_{\xi(y)}$, and then we define the sequence $z(k)$ recursively so that $z(1) = y(1)$ and $z(k+1)$ is the successor of the larger of $y(k+1)$ and $z(k)$. Then z is a strictly increasing sequence in x. Hence we can define $F_y = E_k$ if $y = z(k)$ and $F_y = \emptyset$ otherwise, so $F_y \in \mathcal{C}_{\xi(y)}$ for all $y \in x$ and $E = \bigcup_{y \in x} F_y$, proving the opposite inclusion. □

6.3. *If $\mathcal{C} \subset \mathcal{B}$ and $\mathcal{C}_{\xi(x)} \supset \mathcal{G}$ for some countable ordinal x then $\mathcal{B} = \bigcup_{x<\Omega} \mathcal{C}_{\xi(x)}$.*

Proof. Let $\mathcal{B}_0 = \bigcup_{x<\Omega} \mathcal{C}_{\xi(x)}$. Then $\mathcal{G} \subset \mathcal{B}_0 \subset \mathcal{B}$. Suppose $E_k \in \mathcal{B}_0$ for $k \in \mathbb{P}$. For each k select $y(k) < \Omega$ so that $E_k \in \mathcal{C}_{\xi(y(k))}$ and let $x = \bigcup_{k \in \mathbb{P}} y(k)$. Then x is countable, so $x < \Omega$, and $E_k \in \mathcal{C}_{\xi(x)}$. for all k. Hence both $\bigcup_{k \in \mathbb{P}} E_k$ and $\bigcap_{k \in \mathbb{P}} E_k$ are in $\mathcal{C}_{\xi(s(x))} \subset \mathcal{B}_0$. From 4.2 we have $\mathcal{B}_0 = \mathcal{B}$. □

Theorem 6.4. *In the reals, if $x < y < \Omega$ then $\mathcal{G}_{\xi(x)} \neq \mathcal{G}_{\xi(y)}$.*

Proof. If $\mathcal{G}_{\xi(x)} = \mathcal{G}_{\xi(y)}$ then $\mathcal{H} = \mathcal{G}_{\xi(x)}$ satisfies $\mathcal{G} \subset \mathcal{H} \subset \mathcal{B}$ and $\mathcal{H}_\sigma \cup \mathcal{H}_\delta \subset \mathcal{H}_{\delta\sigma} = \mathcal{G}_{\xi(s(x))} \subset \mathcal{G}_{\xi(y)} = \mathcal{H}$, so $\mathcal{H} = \mathcal{B}$. Therefore in order to prove the theorem we only have to show that $\mathcal{G}_{\xi(x)} \neq \mathcal{B}$.

Now consider the set X constructed in section 5. From 4.5 we have $E \in \mathcal{G}(\mathbb{R})_{\xi(x)}$ if and only if $E' = E \cap X$ is in $\mathcal{G}(X)_{\xi(x)}$, so we only have to show that $\mathcal{G}(X)_{\xi(x)} \neq \mathcal{B}(X)$.

Next consider the collection $\mathcal{C}$ and the sets $R = R_{\sigma\xi(x)}$ and $Q = X \setminus R = Q_{\sigma\xi(x)}$ provided by 5.2. Then $R \notin \mathcal{C}_{\sigma\xi(x)}$, and $\mathcal{C}_{\sigma\xi(x)} = (\mathcal{C}_\sigma)_{\xi(x)} \supset \mathcal{G}(X)_{\xi(x)}$ since $\mathcal{C}_\sigma \supset \mathcal{G}(X)$, so $R \notin \mathcal{G}(X)_{\xi(x)}$. On the other hand, R is in $\mathcal{B}(X)$ since its complement Q is in $\mathcal{C}_{\sigma\sigma\xi(x)} \subset \mathcal{B}(X)$. □

Here's a more general version:

6.5. *Suppose $\mathcal{D}$ and $\mathcal{E}$ are collections of Borel sets in $\mathbb{R}$ with $\mathcal{D}$ countable, and suppose there are countable ordinals d and e so that $\mathcal{E} \subset \mathcal{D}_{\xi(d)}$ and $\mathcal{G} \subset \mathcal{E}_{\xi(e)}$. If $x < y < \Omega$ then $\mathcal{E}_{\xi(x)} \neq \mathcal{E}_{\xi(y)}$.*

7. Suslin Sets are Measurable

Recall that $\zeta = \{ Z_a : a \in \mathbb{P}^\mathbb{P} \}$ where $Z_a = \{ \langle a_1, a_2, \ldots, a_k \rangle : k \in \mathbb{P} \}$. For a fixed sequence $h \in \mathbb{P}^\mathbb{P}$ define $\eta(h) = \{ Z_a : a \in \mathbb{P}^\mathbb{P}$ and $a_k \leq h_k$ for all $k \in \mathbb{P} \}$. Further, for $n \in \mathbb{P}$, define $Z_a(n) = \{ \langle a_1, a_2, \ldots, a_k \rangle : k \leq n \}$ and $\eta(h, n) = \{ Z_a(n) : a \in \mathbb{P}^\mathbb{P}$ and $a_k \leq h_k$ for all $k \leq n \}$. Since we get the same collection by requiring $a \in \mathbb{P}^n$ instead of $a \in \mathbb{P}^\mathbb{P}$, each $\eta(h, n)$ is actually a finite collection of finite sets. Hence

7.1. *For any $C \colon \mathbb{P}^+ \to \mathcal{C}$, $\Phi_{\eta(h,n+1)} \subset \Phi_{\eta(h,n)} \in \mathcal{C}_{\delta\sigma}$.*

7.2. *For any $C \colon \mathbb{P}^+ \to \mathcal{C}$,*

$$\Phi_{\eta(h)}(C) = \bigcap_{n \in \mathbb{P}} \Phi_{\eta(h,n)}(C).$$

Hence $\mathcal{C}_{\eta(h)} \subset \mathcal{C}_{\delta\sigma\delta}$.

Proof. It is immediate that $\Phi_{\eta(h)}(C) \subset \Phi_{\eta(h,n)}(C)$ for each n. Now suppose $x \in \Phi_{\eta(h,n)}(C)$ for all n. Then for each n there is a sequence a_n so that $a_{nk} \leq h_k$ for all $k \leq n$ and $x \in C_{a_{n1}a_{n2}\ldots a_{nk}}$ for all $k \leq n$. Then $x \in C_{a_{n1}}$ for all n. Hence there is an infinite set $I_1 \subset \mathbb{P}$ and $b_1 \in \mathbb{P}$ so that $b_1 = a_{n1}$ for all $n \in I_1$. Now continue recursively, defining an infinite set $I_r \subset I_{r-1}$ and b_r so that $b_1 b_2 \ldots b_r = a_{n1} a_{n2} \ldots a_{nr}$ for all $n \in I_r$ and $x \in C_{b_1 b_2 \ldots b_r}$. Then $b_r \leq h_r$ for all r so $Z_b \in \eta(h)$, and thus $x \in \Phi_{\eta(h)}$. □

Next we define $\theta(h, n) = \{ Z_a : a \in \mathbb{P}^\mathbb{P}$ and $a_k \leq h_k$ for all $k \leq n \}$ and $\theta(h, n, k) = \{ Z_a \in \theta(h, n) : a_{n+1} = k \}$. We allow $n = 0$ in these definitions.

7.3. *For any $C \colon \mathbb{P}^+ \to \mathcal{C}$, $\Phi_\zeta(C) = \Phi_{\theta(h,0)}(C)$, $\Phi_{\theta(h,n,k)}(C) \subset \Phi_{\theta(h,n,k+1)}(C)$ and*

$$\Phi_{\theta(h,n+1)}(C) = \bigcup_{k \in \mathbb{P}} \Phi_{\theta(h,n,k)}(C).$$

Now consider an outer measure μ^* defined on $\mathcal{P}(X)$. The Carathéodory construction produces a measure μ on a σ-algebra $\mathcal{M}$. Here $\mathcal{M}$ is defined as the collection of those sets E satisfying the measurability condition

$$\mu^*(T) = \mu^*(T \cap E) + \mu^*(T \setminus E)$$

for all $T \subset X$, and μ is just the restriction of μ^* to $\mathcal{M}$. We need two facts about this measure. First,

7.4. *For any $T \subset X$ and $E \in \mathcal{M}$ define $\mu_T(E) = \mu^*(T \cap E)$. Then μ_T is a measure.*

Proof. All we need is countable additivity. Suppose $E_k \in \mathcal{M}$ are pairwise disjoint. Applying the measurability condition to E_1 and $T \cap (E_1 \cup E_2)$ proves $\mu_T(E_1 \cup E_2) = \mu_T(E_1) + \mu_T(E_2)$. By induction we have $\mu_T\left(\bigcup_{k=1}^n E_k\right) = \sum_{k=1}^n \mu_T(E_k)$ for all $n \in \mathbb{P}$. Hence $\sum_{k=1}^\infty \mu_T(E_k) = \lim_{n\to\infty} \mu_T\left(\bigcup_{k=1}^n E_k\right) \le \mu_T\left(\bigcup_{k=1}^\infty E_k\right)$. The opposite inequality comes from countable sub-additivity of μ^*. □

7.5. *For any subsets T and S of X define $\bar{\mu}_T(S) = \inf\{\, \mu_T(E) : E \in \mathcal{M}$ and $E \supset S \,\}$.*

1. *$\mu^*(T \cap S) \le \bar{\mu}_T(S)$.*
2. *There is $\bar{S} \in \mathcal{M}$ with $\bar{S} \supset S$ and $\bar{\mu}_T(S) = \mu_T(\bar{S})$.*
3. *If V is an increasing sequence of subsets of X then $\bar{\mu}_T\left(\bigcup_{k=1}^\infty V_k\right) = \lim_{n\to\infty} \bar{\mu}_T(V_n)$.*

Proof. Part 1 is obvious from the definition. For part 2, if $\bar{\mu}_T(S) = \infty$ just use $\bar{S} = X$. Otherwise find $E_k \in \mathcal{M}$ with $\mu_T(E_k)$ finite, $E_k \supset E_{k+1} \supset S$ and $\lim_{n\to\infty} \mu_T(E_n) = \bar{\mu}_T(S)$ and set $\bar{S} = \bigcap_{k=1}^\infty E_k$.

Now write $S = \bigcup_{k=1}^\infty V_k$ and select $\bar{S}$ and $\bar{V}_k$ by part 2. Define $\bar{V}'_k = \left(\bigcap_{j=k}^\infty \bar{V}_j\right) \cap \bar{S}$, so $\bar{V}'_k \subset \bar{V}'_{k+1}$, and define $\bar{S}' = \bigcup_{k=1}^\infty \bar{V}'_k$. Note that part 2 is still satisfied for $\bar{V}'_k$ and $\bar{S}'$. Hence $\bar{\mu}_T\left(\bigcup_{k=1}^\infty V_k\right) = \bar{\mu}_T(S) = \mu_T(\bar{S}') = \mu_T\left(\bigcup_{k=1}^\infty \bar{V}'_k\right) = \lim_{n\to\infty} \mu_T(\bar{V}'_n) = \lim_{n\to\infty} \bar{\mu}_T(V_n)$. □

Theorem 7.6. *Suppose that the measure μ with domain $\mathcal{M}$ is obtained from the outer measure μ^* by the Carathéodory construction. Then $\mathcal{M}_\zeta \subset \mathcal{M}$.*

Proof. Let $C\colon \mathbb{P}^+ \to \mathcal{M}$ and let $E = \Phi_\zeta(C)$. To show that E is measurable we need to show $\mu^*(T) \ge \mu^*(T \cap E) + \mu^*(T \setminus E)$ since the opposite inequality follows from countable sub-additivity. There is nothing to prove if $\mu^*(T) = \infty$ so we assume $\mu^*(T)$ is finite.

Let $\epsilon > 0$ be chosen. From 7.3 and 7.5.3 we define a sequence h recursively so that

$$\bar{\mu}_T\left(\Phi_{\theta(h,n)}(C)\right) \ge \bar{\mu}_T\left(\Phi_{\theta(h,n-1)}(C)\right) - \frac{\epsilon}{2^n}$$

for all $n \ge 1$. Remember that $\Phi_{\theta(h,0)}(C) = E$. Hence, adding the inequalities,

$$\bar{\mu}_T\left(\Phi_{\theta(h,n)}(C)\right) \ge \bar{\mu}_T(E) - \epsilon \ge \mu^*(T \cap E) - \epsilon.$$

Now $E_n = \Phi_{\eta(h,n)}(C)$ is in $\mathcal{M}_{\delta\sigma\delta} \subset \mathcal{M}$ and $E_n \supset \Phi_{\theta(h,n)}(C)$. Hence $\mu^*(T \cap E_n) \ge \bar{\mu}_T\left(\Phi_{\theta(h,n)}(C)\right) - \epsilon$ so $\mu^*(T) = \mu^*(T \cap E_n) + \mu^*(T \setminus E_n) \ge \mu^*(T \cap E) + \mu^*(T \setminus E_n) - \epsilon$. But $E_n \supset E_{n+1}$ and $\bigcap_{k=1}^\infty E_k \subset E$ by 7.2, so $\lim_{n\to\infty} \mu^*(T \setminus E_n) = \lim_{n\to\infty} \mu_T(X \setminus E_n) = \mu_T\left(\bigcup_{k=1}^\infty (X \setminus E_k)\right) = \mu_T\left(X \setminus \bigcap_{k=1}^\infty E_k\right) \ge \mu_T(X \setminus E) = \mu^*(T \setminus E)$. Hence we have $\mu^*(T) \ge \mu^*(T \cap E) + \mu^*(T \setminus E) - \epsilon$. We finish the proof by letting $\epsilon \to 0$. □

Corollary 7.7. *For $\mathcal{M}$ as in Theorem 7.6, if $\mathcal{G} \subset \mathcal{M}$ then $\mathcal{Z} \subset \mathcal{M}$.*

References

1. Felix Hausdorff, *Set theory*, third ed., Chelsea, New York, 1957, translated by John Aumann, *et. al.*
2. Stanisław Saks, *Theory of the integral*, second revised ed., Hafner, New York, 1937, translated by L. C. Young.
3. Michael Suslin, *Sur une définition des ensembles mesurables B sans nombres transfinis*, C. R. Acad. Sci. Paris **164** (1917), 88–91.