MR1275735 (95e:05009) 05A17 (05A10)
Simion, Rodica (1-GWU)
Combinatorial statistics on noncrossing partitions. (English summary)
J. Combin. Theory Ser. A **66** (1994), *no. 2,* 270–301.

Summary: "Four statistics, ls, rb, rs, and lb, previously studied on all partitions of $\{1, 2, \cdots, n\}$, are applied to noncrossing partitions. We consider single and joint distributions of these statistics and prove equidistribution results. We obtain q- and p, q-analogues of Catalan and Narayana numbers which refine the rank symmetry and unimodality of the lattice of noncrossing partitions. Two unimodality conjectures, one of which pertains to Young's lattice, are stated. We exhibit relations between statistics on noncrossing partitions and established permutation statistics applied to restricted permutations. All our proofs are combinatorial, relying on the construction of bijective correspondences and on structural properties of the lattice of noncrossing partitions."

Reviewed by *Wen Li Chen*

MR1264480 (95f:05012) 05A18 (06A07 06B99)
Edelman, Paul H. (1-MN); **Simion, Rodica** (1-GWU)
Chains in the lattice of noncrossing partitions. (English summary)
Discrete Math. **126** (1994), *no. 1-3,* 107–119.

A partition of the set $\mathbf{n} = \{0, 1, \cdots, n\}$ is called noncrossing if for every $a < b < c < d$, if a and c are in the same block and b and d are in the same block then all four are in the same block. The set $\mathrm{NC}(\mathbf{n})$ of noncrossing partitions of $\mathbf{n}$, partially ordered by refinement, is a lattice.

Suppose that π' covers π in $\mathrm{NC}(\mathbf{n})$. Then π' may be obtained from π by merging two blocks B_1 and B_2 of π. Let m_1 and m_2 be the least elements of B_1 and B_2 and define $\lambda(\pi, \pi')$ to be the larger of m_1 and m_2. Then λ is an R-labeling of $\mathrm{NC}(\mathbf{n})$, that is, a labeling of the covering relations such that if $x < y$ then there is a unique saturated chain from x to y along which the labels are increasing.

The R-labeling λ has the property that the labels along each maximal chain form a permutation of $[n] = \{1, 2, \cdots, n\}$. For any permutation σ of $[n]$, let $m(\sigma)$ be the number of maximal chains whose label sequence is σ. The authors give a recurrence for $m(\sigma)$ and show that it is equal to the number of noncrossing partitions of $[n]$ each of whose blocks is a decreasing subsequence of σ. From this it follows that the Möbius function of $\mathrm{NC}(\mathbf{n})$ is $(-1)^{n-1}C_{n-1}$, where C_{n-1} is a Catalan number.

Next the authors consider connections with the weak (Bruhat) order on permutations. They show that if $\sigma < \sigma'$ in the weak order then $m(\sigma) < m(\sigma')$. Finally, they prove some formulas involving Möbius inversion.

Some analogous results for the lattice of all partitions of a set are also discussed.

Reviewed by *Ira Gessel*

MR1144402 (92j:06003) 06A07 (05A18 05D05)
Simion, Rodica (1-GWU); **Ullman, Daniel** (1-GWU)
On the structure of the lattice of noncrossing partitions.
Discrete Math. **98** (1991), *no. 3,* 193–206.

A partition of $\{1, \cdots, n\}$ is called noncrossing if whenever $1 \le a < b < c < d \le n$ with a, c in the same block, and b, d in the same block, then all four elements are in the same block. With the refinement order the set of noncrossing partitions becomes a lattice, denoted by $\mathrm{NC}(n)$. Via an order-reversing involution it is proved that $\mathrm{NC}(n)$ is self-dual. Two proofs are given for the SCD property, i.e., for the existence of a symmetric chain decomposition of $\mathrm{NC}(n)$. The first one is recursive and uses the known fact that the SCD property is preserved under the poset product operation. The second one is constructive and uses the parenthesization method. Finally, through counting arguments based on the symmetric chain decomposition, several identities which involve Catalan numbers are derived.

Reviewed by *Konrad Engel*

MR710893 (84j:05014) 05A15 (05A17 05C05)

Prodinger, Helmut (A-TUWN-B)

A correspondence between ordered trees and noncrossing partitions.

Discrete Math. **46** (1983), *no. 2*, 205–206.

The Narayana numbers $(1/n)\binom{n}{k}\binom{n}{k-1}$ count the ordered trees with n edges and k leaves and the noncrossing partitions of $\{1,2,\cdots,n\}$ into k blocks (a partition is noncrossing if the existence of four numbers $a<b<c<d$ such that a and c are in one block and b and d are in another block is forbidden). Given such a tree, it is shown in this note that labeling the edges in preorder, and then reading the chain as follows: read the longest chain containing the largest element, delete it, and continue the process until all edges are deleted, results in a corresponding sequence. The reverse transformation is also described.

Reviewed by *S. Zaks*

MR676723 (83m:05048) 05C05 (05A17 06A10)

Edelman, Paul H.

Multichains, noncrossing partitions and trees.

Discrete Math. **40** (1982), *no. 2-3*, 171–179.

A partition $X=\{B_1,B_2,\cdots,B_k\}$ of the set $\{1,2,\cdots,m\}\equiv[m]$ is said to be noncrossing if there do not exist four numbers $a<b<c<d$ such that $a,c\in B_i$ and $b,d\in B_j$ where $i\neq j$. Let T_m denote the lattice of all noncrossing partitions of $[m]$ ordered by refinement. A k-ary tree is defined to be an ordered rooted tree in which each vertex has at most k subtrees and, furthermore, there are fewer than k subtrees when one distinguishes between the 1st, 2nd,$\cdots$, and kth subtree. The author establishes a bijection between the k-ary trees with m vertices and the number of multichains in T_m of cardinality $k-1$.

Reviewed by *J. W. Moon*

MR583216 (81i:05018) 05A17
Edelman, Paul H.
Chain enumeration and noncrossing partitions.
Discrete Math. **31** (1980), *no. 2*, 171–180.

A partition $B_1, B_2, \cdots, B_k$ of the set $[m] = \{1, 2, \cdots, m\}$ is said to be noncrossing (n.c.) if there do not exist $a < b < c < d$ such that $a, c \in B_i$ and $b, d \in B_j$ and $i \neq j$. Let T_m be the set if all n.c. partitions of $[m]$ ordered by refinement. G. Kreweras [same journal **1** (1972), no. 4, 333–350; MR0309747 (46 #8852)] showed that T_m is a ranked lattice.

Let $0 < t_1 < \cdots < t_r < m-1$ and let $N(t_1, \cdots, t_r)$ denote the number of chains $0 < x_1 < \cdots < x_r < 1$ in T_m such that $\operatorname{rank}(x_i) = t_i$. Stanley conjectured and it is here proven that

$$N(t_1, \cdots, t_r) = \frac{1}{m}\binom{m}{t_1}\binom{m}{t_2 - t_1}\cdots\binom{m}{m-1-t_r}.$$

An n.c. partition of $[km]$ is said to be k-divisible if the cardinality of every block is a multiple of k. A formula similar to the one above is found for the ranked P.O. set of such partitions. The concept of n.c. is also generalized to 2-partitions and again a similar formula is given. For each of the above the zeta polynomials are calculated.

Reviewed by *Daniel I. A. Cohen*