MATH 503: SOLUTIONS TO PROBLEM SET 12

1. Problems From Section 5.5

Problem (6)**.** Let K be a cyclic group, H an arbitrary group, and ϕ_1, ϕ_2 homomorphisms from K into $\operatorname{Aut}(H)$ such that $\phi_1(K)$ and $\phi_2(K)$ are conjugate subgroups of $\operatorname{Aut}(H)$. If K is infinite, assume that ϕ_1 and ϕ_2 are injective. Prove by constructing an explicit isomorphism that $H \rtimes_{\phi_1} K \cong H \rtimes_{\phi_2} K$.

Solution. Let $\sigma \in \operatorname{Aut}(H)$ be an element that does the conjugation of the hypothesis, i.e. suppose $\sigma\phi_1(K)\sigma^{-1} = \phi_2(K)$. Since K is cyclic, the induced automorphism of K is raising to some power. (We know the automorphism groups of cyclic groups, remember?) Thus for some integer a, we have $\sigma\phi_1(k)\sigma^{-1} = \phi_2(k)^a$ for all $k \in K$. We now define $\psi\colon H \rtimes_{\phi_1} K \to H \rtimes_{\phi_2} K$ by $\psi((h, k)) = (\sigma(h), k^a)$.

We claim that ψ is a homomorphism. To see this, we compute

$$\begin{aligned}
\psi((h,k)(h',k')) &= \psi((h\phi_1(k)(h'), kk')) \\
&= (\sigma(h\phi_1(k)(h')), (kk')^a) \\
&= (\sigma(h\phi_1(k)(h')), k^a k'^a) \\
&= (\sigma(h)\sigma(\phi_1(k)(h')), k^a k'^a) \\
&= (\sigma(h)\sigma(\phi_1(k)(h')), k^a k'^a) \\
&= (\sigma(h)\phi_2(k)^a(\sigma(h')), k^a k'^a) \\
&= (\sigma(h)\phi_2(k^a)(\sigma(h')), k^a k'^a) \\
&= (\sigma(h), k^a)(\sigma(h'), k'^a) \\
&= \psi((h,k))\psi((h',k'))
\end{aligned}$$

We can construct a 2-sided inverse ψ' for ψ as follows: as in the previous paragraph, there is an integer b such that $\sigma^{-1}\phi_2(k)\sigma = \phi_1(k)^b$ for all $k \in K$. We further have $k^{ab} = k$ for all $k \in K$. It is now straightforward to verify that $\psi'\psi$ $\psi\psi'$ are the identity maps of $H \rtimes_{\phi_1} K$ and $H \rtimes_{\phi_2} K$.

□

Problem (10)**.** I really can't be bothered to retype this one.

Solution. Let G be a group of order 147.

(a) We have $147 = 3 \cdot 7^2$, so by the Fundamental Theorem of Finite Abelian Groups, an abelian G is $C_3 \times C_{49}$ or $C_3 \times C_7 \times C_7$.

(b) The number of Sylow 7-subgroups of G, n_7, satisfies $n_7 \equiv 1 \mod 7$ and $n_7 \mid 3$. It follows that $n_7 = 1$, so G has a unique Sylow 7-subgroup which is therefore normal.

(c) Let H be the Sylow 7-subgroup of G and assume $H \cong C_{49}$. Let $K \in \operatorname{Syl}_3(G)$. Then Theorem 12 tells us that G is the semidirect product $H \rtimes_\phi K$, and since G is not abelian by hypothesis, ϕ is nontrivial. We know that $|K| = 3$, so K is cyclic, and $|\operatorname{Aut}(H)| = 42 = 3 \cdot 2 \cdot 7$, so the image $\phi(K)$

Date: March 24, 2004.

is a Sylow 3-subgroup of $\mathrm{Aut}(H)$, and any two such are conjugate. Exercise 6 now shows that such a semidirect product is unique up to isomorphism.

(d) Since $2^3 \equiv 1 \mod 7$, we see that t_1 and t_2 have order 3. They commute, so $P \cong C_3 \times C_3$. By the order formula for general linear groups, $|GL_2(\mathbb{F}_7)| = 48 \cdot 42 = 3^2 \cdot 2^5 \cdot 7$, so P is a Sylow 3-subgroup of $GL_2(\mathbb{F}_7)$. By the second Sylow theorem, any 3-subgroup of $GL_2(\mathbb{F}_7)$, in particular any subgroup of order 3, is conjugate to a subgroup of P.

(e)

$$G_1 = \langle x, y, t \mid x^7 = y^7 = t^3 = 1, txt^{-1} = x^2, tyt^{-1} = y\rangle$$
$$G_2 = \langle x, y, t \mid x^7 = y^7 = t^3 = 1, txt^{-1} = x, tyt^{-1} = y^2\rangle$$
$$G_3 = \langle x, y, t \mid x^7 = y^7 = t^3 = 1, txt^{-1} = x^2, tyt^{-1} = y^2\rangle$$
$$G_4 = \langle x, y, t \mid x^7 = y^7 = t^3 = 1, txt^{-1} = x^2, tyt^{-1} = y^4\rangle$$

There is an isomorphism $G_1 \to G_2$ defined by $x \mapsto y, y \mapsto x, t \mapsto t$.

(f) From the relations $y \in Z(G_1)$, so $|Z(G_1)| \geq 7$. To see that G_3 and G_4 have trivial centers, note that from the definition of semidirect product, any element of either group may be expressed uniquely as $g = x^i y^j t^k$, where $0 \leq i < 7$, $0 \leq j < 7$, and $0 \leq k < 3$. If $g \in Z(G_r)$ for either group ($r = 3, 4$), we have $tgt^{-1} = g$. For $r = 3$ we get $tgt^{-1} = x^{2i}y^{2j}t^k = x^i y^j t^k$ and so $i = j = 0$. For $r = 4$ we get $tgt^{-1} = x^{2i}y^{4j}t^k = x^i y^j t^k$ and so $i = j = 0$. Thus the hypothetical element of the center is t^k, and it is clear from the relations that no such element can lie in $Z(G_r)$. Thus $Z(G_3)$ and $Z(G_4)$ are trivial. Thus G_1 is not isomorphic to either G_3 or G_4.

(g) Let C be a subgroup of G_3 of order 7. By the Sylow theorems, C lies in a conjugate of the Sylow 7-subgroup of G_3, so $C \leq Q = \langle x, y\rangle \leq G_3$ since Q is normal. Thus $C = \langle x^i y^j\rangle$. To see that $C \lhd G_3$, it is enough show that C is mapped to itself by conjugation by x, y, and t. The cases of x and y are immediate since x and y commute, while $t(x^i y^j)t^{-1} = x^{2i}y^{2j} = (x^i y^j)^2 \in C$. Thus any subgroup of G_3 of order 7 is normal. This is not true in G_4, since $C' = \langle xy\rangle \leq Q' \leq G_4$ is not normal in G_4, since $txyt^{-1} = x^2y^4 \notin C$. It follows that G_3 and G_4 are not isomorphic.

(h) To complete the classification, it is enough to assume that G is not abelian and that the Sylow 7-subgroup $Q \leq G$ is not cyclic, and show that G is isomorphic to one of G_1, G_3, or G_4. Under these hypotheses, we have $Q \cong C_7 \times C_7$, and as in (c) letting $K \in \mathrm{Syl}_3(G)$ we have $G \cong Q \rtimes_\phi K$. By Exercise 6, such a group is determined by the conjugacy class of $\phi(K)$ in $\mathrm{Aut}(H)$. This yields at most 4 possibilities (i.e. $Q \rtimes_\phi K \cong G_1, G_2, G_3,$ or G_4) , as seen in (e); we saw in (f) and (g) that there are only three distinct groups of this type. In other words, $Q \rtimes_\phi K \cong G_1, G_3,$ or G_4. □

2. Problems From Section 6.1

Problem. Prove that if P is a non-abelian group of order p^3 then $|Z(P)| = p$ and $P/Z(P) \cong C_p \times C_p$

Proof. Since P is a p-group, it has a nontrivial center, and since P is not abelian, $|Z(P)| \neq p^3$. If $|Z(P)| = p^2$, then $P/Z(P)$ has order p, so it is cyclic, and P is abelian, a contradiction. Therefore $|Z(P)| = p$ and $P/Z(P)$ is isomorphic to either C_{p^2} or $C_p \times C_p$. In the first case $P/Z(P)$ is cyclic, so P is abelian, a contradiction. Thus $P/Z(P) \cong C_p \times C_p$. □

3. Problems From Section 6.2

Problem. In the group $S_3 \times S_3$ exhibit a pair of Sylow 2-subgroups that intersect in the identity and a pair that intersect in a group of order 2.

Solution. Let $P_1 = \langle(12), (45)\rangle$, $P_2 = \langle(12), (56)\rangle$, and $P_3 = \langle(23), (56)\rangle$. Then $P_1 \cap P_3 = 1$, while $|P_1 \cap P_2| = 2$. □

Problem. Prove that if $|G| = 380$ then G is not simple.

Solution. Note that $380 = 2^2 \cdot 5 \cdot 19$. If G is simple, then $n_{19} \neq 1$, $n_5 \neq 1$. From the Sylow theorems, we have $n_{19} \geq 20$ and $n_5 \geq 6$, so G has at least $20(19-1)+6(5-1) > 380$ elements, a contradiction. □

4. Problems From Section 6.3

Problem. Prove that the commutator subgroup of the free group F_2 on two generators is not finitely generated.

Solution 1: topology. Subgroups of F_2 correspond to covering spaces of the figure-8 graph, and the commutator subgroup corresponds to the covering graph which is $\cup_{n\in\mathbb{Z}}\{x = n\} \cup \cup_{m\in\mathbb{Z}}\{y = m\}$. The commutator subgroup can be recovered as the fundamental group of this graph, which is free with one generator for each edge not in a maximal tree. A maximal tree in this graph is $\{x = 0\} \cup \cup_{m\in\mathbb{Z}}\{y = m\}$, which leaves out infinitely many edges, so the commutator subgroup is not finitely generated. □

Solution 2: algebra (from M. Mazur). Let $H = \oplus_{n\in\mathbb{Z}} C_2$, let $C = \langle t\rangle$ be an infinite cyclic group, and let $\phi\colon C \to \mathrm{Aut}(H)$ be the homomorphism that sends t to the "right shift" automorphism. Let $G = H \rtimes_\phi C$ and note that G is generated by t and the vector $(\ldots, 0, 0, \ldots, 0, 1, 0, \ldots, 0, 0, \ldots)$. Note that G' is the subgroup of H where the entries sum to zero; this sum makes sense since only finitely many entries in the infinite vector are nonzero. Observe that G' is not finitely generated.

If F_2' were finitely generated, any group G generated by 2 elements would have a finitely generated commutator subgroup, since G' would be the image of F_2'. Since our G of the previous paragraph has the property that G' is not finitely generated, F_2' is not finitely generated either. □

Mathematics Department, Binghamton Univeristy, P. O. Box 6000, Binghamton, New York, 13902-6000
E-mail address: dikran@math.binghamton.edu