RESEARCH STATEMENT (extended)

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1 Summary

My research interests concentrate mostly on combinatorial and geometric group theory. Most recently I was involved in studying geometry of the outer automorphisms group $Out(F_n)$ of a free group F_n. Another big part of my research interests lies in the area of groups acting on rooted trees by automorphisms and, in particular, generated by automata (or self-similar groups). I am also interested in questions related to Thompson's group F and in geometry of the spaces F acts on. Furthermore, I am a co-author of a package for computations in groups generated by autoamta and I am highly interested in developing of a software in group theory.

In the subsequent sections I briefly describe my major results and future plans.

2 Geometry of $Out(F_n)$

Let $Out(F_n)$ denote the group of outer automorphisms of the free group F_n of rank $n > 2$. There is a strong analogy between $Out(F_n)$ and the mapping class group of a surface on the one hand and arithmetic groups on the other, which has been a driving force for the research in these areas for the last couple of decades. One of the first examples of this analogy was the foundational paper of Culler and Vogtmann [11], which introduced Outer Space, the analogue for $Out(F_n)$ of Teichmüller space for the mapping class group and of symmetric spaces for arithmetic groups. The work that followed has yielded numerous statements about the topological and (co)homological properties of $Out(F_n)$ and the spaces upon which it acts.

While the topology of Outer Space is well-understood, its geometry is not. In contrast, the geometries of Teichmüller space and the symmetric spaces are well-studied. One key ingredient for Teichmüller space is the celebrated result of Masur and Minsky, who proved that the curve complex is hyperbolic [19]. This fact has been used, for instance, to prove quasiisometric rigidity of the mapping class group.

In the $Out(F_n)$ world the search for a 'correct' analogue of a curve complex for $Out(F_n)$ is still not over. Many different candidates were proposed and only very recently Bestvina and Feighn in [2] proved that one of these candidates, namely the complex of free factors, is hyperbolic. The vertices of this complex are conjugacy classes of free factors in F_n and the edges are induced by inclusion. On the other hand, as of now there are no applications of this fact for $Out(F_n)$ similar to mapping class groups world.

My recent work [24] (joint with Lucas Sabalka) deals mainly with another proposed analog of the curve complex defined by Kapovich and Lustig in [18]: the edge splitting complex $\mathcal{ES}_n$. The vertices of $\mathcal{ES}_n$ correspond to conjugacy classes of free factorizations of F_n into two nontrivial free factors. Two vertices of this complex are connected with an edge if there exists a free factorization in each conjugacy class such that the two factorizations have a common refinement.

The main result of [24] is:

Theorem 1. *The space $\mathcal{ES}_n$ is not Gromov hyperbolic.*

This theorem shows that the space $\mathcal{ES}_n$ is not the curve complex analogue we seek to better understand the geometry of $Out(F_n)$. It is a consequence of a more general statement – namely:

Theorem 2. *For $n > 2$, the space $\mathcal{ES}_n$ contains a quasiisometrically embedded copy of $\mathbb{R}^m$ for every $m \geq 1$.*

Our proof relies on attaining an understanding of distances in $\mathcal{ES}_n$ in purely combinatorial terms and uses a new technique based on the notion of i-length which is itself based roughly on having many subwords of elements of the basis with complicated Whitehead graphs.

A further corollary of Theorem 2 is:

Theorem 3. *The space $\mathcal{ES}_n$ has infinite asymptotic dimension. The dimension of every asymptotic cone of $\mathcal{ES}_n$ is infinite.*

To my knowledge, this is the only naturally defined space with these properties and with a natural cocompact group action of a group which is not known to have infinite asymptotic dimension.

3 Subsets of bases in relatively free groups

During working on the paper [24] me and Lucas Sabalka came across an interesting question, whose simplest formulation is as follows. Given a free group F with a fixed basis $A = \{a_1, \ldots, a_n\}$ suppose that a subset S of F is primitive in F (i.e., is a part of some basis of F) and is such that its elements expressed in terms of basis A do not involve generators $a_l, \ldots, a_n$. Is it always true that S will also be primitive in $\hat{F} = F(\{a_1, \ldots, a_{l-1}\})$?

We answered this question and generalized it to several relatively free groups. Namely, we proved the following theorem:

Theorem 4. *Let $S \subset \hat{F}$ be primitive in F mod V, where V is a fully invariant subgroup of F. Denote by $\hat{V} = V \cap \hat{F}$ – a fully invariant subgroup of $\hat{F}$.*

1. *If $F/V = F$ is free, then S is primitive in $\hat{F}$.*
2. *If F/V is free nilpotent, then S is primitive in $\hat{F}$ mod $\hat{V}$.*
3. *If F/V is free solvable, then S is primitive in $\hat{F}$ mod $\hat{V}$.*

Our proof for the second and the third case relies on the first case, which, in turn, is based on primitivity criterion for subsets of free groups that uses Fox Calculus proved by Birman in [3] and Umirbaev in [33]. An interesting question is whether there is a variety of groups where Theorem 4 does not hold.

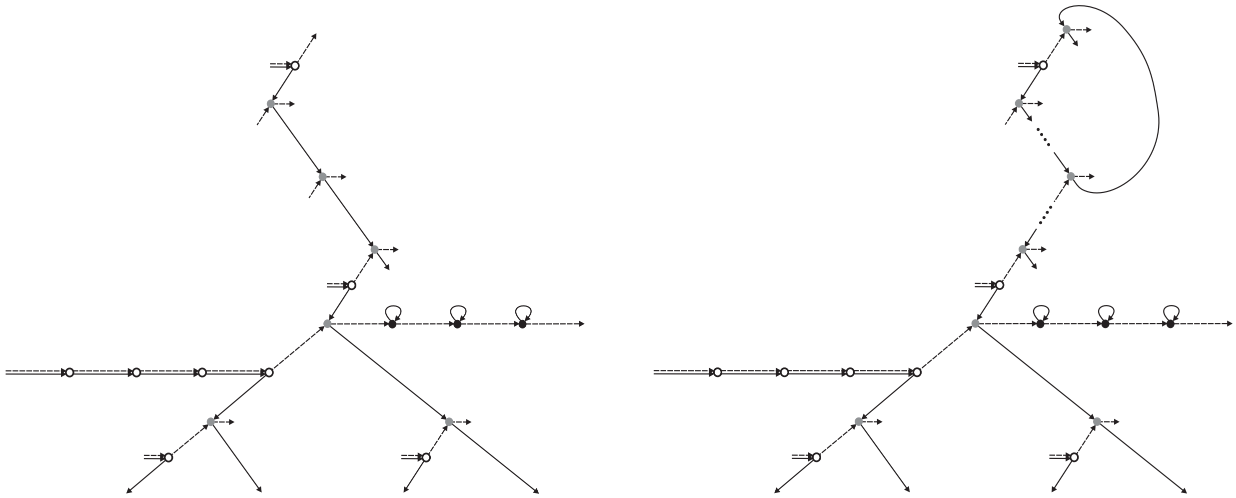

Figure 1: Schreier graphs of Thompson's group F

4 Thompson's group F

Another area of my research is related to Thompson's group F. This group can be defined as a group of all piecewise linear orientation preserving homeomorphisms of the unit interval $[0,1]$ into itself, whose break points are dyadic rational numbers and the slopes are powers of 2. This group was first considered by R. Thompson in 1965.

In papers [25] and [26] I completely described Schreier graphs of the action of F on the orbits of all points of the interval $(0,1)$, classified them up to isomorphism and showed that they have trivial automorphism groups.

Theorem 5. *The Schreier graphs of Thompson's group F on the orbits of points in $(0,1)$ are depicted in Figure 1 (the left and right sides represents Schreier graphs of irrational and rational numbers, respectively).*

The studying of the Schreier graphs of F was partially inspired by the question of amenability of F – one of the most intriguing wide open questions about this group. In particular, if some Schreier graph is non-amenable the whole group F would be non-amenable. Even though all constructed graphs turned out to be amenable, the knowledge about the structure of Schreier graphs provides some additional information about F itself. Schreier graphs appear naturally in many contexts. For example, recent papers [12] and [4] describe Schreier graphs of the actions of certain groups generated by automata on the boundary of the rooted tree.

Furthermore, in [26] I study Schreier dynamical systems arising from the action of F on the closure of the space of pointed Schreier graphs by shifting the basepoint. It turns out that the action of F on $[0,1]$ is completely recovered from the action on each orbit:

Theorem 6. *For each $x \in (0,1)$, the action of F on the perfect kernel of the Schreier dynamical system associated to the orbit of x is topologically conjugate to the standard action of F on the Cantor set X^ω.*

Another result of [25] is related to the positive monoid P of F. Grigorchuk and Stepin reduced the question of amenability of F to the right amenability of P. Moreover, I showed that the amenability of F is equivalent to the amenability of the induced subgraph Γ_P of the Cayley graph

Γ_F of F containing the positive monoid P. In [25] I constructed the induced subgraph Γ_S of Γ_F containing all the vertices of the form $x_n v$ for $n \geq 0, v \in \{x_0, x_1\}^*$ and proved that this graph is non-amenable. Certain generalization of my results related to the non-amenability of induced subgraphs of P are obtained by Staley in [29].

Groups generated by automata

The first appearance of these groups goes back to the beginning of 1960's. Self-similar groups have many interesting and important properties. The class of self-similar groups contains exotic examples, such as Burnside groups, groups of intermediate growth, as well as familiar examples, such as free groups or free products of finite groups that are well known and are regular objects of study in combinatorial group theory.

It was observed recently that the class of self-similar groups naturally appears in mathematics. The most recent examples come from combinatorics and are related to the famous combinatorial problem known as Hanoi Towers Game [17]. Slightly older examples, due first of all to Nekrashevych [21], are related to holomorphic dynamics and random walks.

5 Iterated monodromy groups

One of the most remarkable discoveries in the recent years is the observation, due to Nekrashevych, that the so-called iterated monodromy groups (IMG), which can be related to any self-covering map, belong to the class of self-similar groups and that, in the most natural situations, there is an explicit procedure representing them by finite automata [21].

In paper [16] (joint with R. Grigorchuk and Z. Šunić) we study the iterated monodromy group $\mathrm{IMG}(z^2+i)$ of the map $z \mapsto z^2+i$, which is generated by 3 nontrivial states of the 4-state automaton over a 2-letter alphabet. We show that this group is a regular branch group, thus presenting an example of a branch group which naturally appears in holomorphic dynamics. The main body of the article is devoted to the calculation of an L-presentation for $\mathrm{IMG}(z^2+i)$ (i.e., a presentation which involves a finite set of relators and their iterations by substitution). More precisely, we proved

Theorem 7. *The group* $\mathrm{IMG}(z^2+1)$ *has the following recursive presentation:*

$$\begin{aligned}\mathrm{IMG}(z^2+i) \cong \big\langle a, b, c \;\big|\; & \phi^n(a^2), \phi^n\big((ac)^4\big), \phi^n([c,ab]^2), \phi^n([c,bab]^2), \\ & \phi^n([c,ababa]^2), \phi^n([c,ababab]^2), \phi^n([c,bababab]^2), n \geq 0\big\rangle,\end{aligned}$$

where ϕ is a substitution defined on words in the free monoid over the alphabet $\{a,b,c\}$ by $\phi(a) = b$, $\phi(b) = c$ and $\phi(c) = aba$.

Although it is known that L-presentations are quite common for groups of branch type the number of examples in which explicit computation is possible is rather small.

The presence of L-presentations is important from different points of view. For example, such presentations can be used to embed a group into a finitely presented group in a way that preserves many properties of the original group. We use the obtained L-presentation of $\mathrm{IMG}(z^2+i)$ to embed it into a finitely presented group with 4 generators and 10 relators, which is amenable but not elementary amenable.

It was shown by K.-U. Bux and R. Pérez that $\mathrm{IMG}(z^2+i)$ has intermediate growth and, hence, is amenable. We find a self-similar measure on $\mathrm{IMG}(z^2+i)$ providing a different proof that the group is amenable.

The self-similar measure is closely related to the problem of computation of the spectrum of a Hecke type operator that can be related to any group acting on a rooted tree and to the problem of computation of the spectrum of the discrete Laplace operator on the boundary Schreier graph of a group. Unfortunately, the spectral problem is not solved yet for $\mathrm{IMG}(z^2+i)$. What we were able to construct is a rational map on $\mathbb{R}^3$ whose proper invariant set gives the spectrum of the Markov operator after intersection by a corresponding line. Further efforts in the description of the shape of the attractor (and hence of the spectrum) should be a task for future research.

6 Groups of intermediate growth

Part of my dissertation is devoted to the study of Sushchansky groups introduced by V. Sushchansky in [32] in 1979 as one of the pioneering examples of Burnside groups. The results of this section are published in [9] (joint with I. Bondarenko). Sushchansky used a different language, namely the language of tableaux, introduced by L. Kaluzhnin to study properties of iterated wreath products. For each prime $p > 2$, V. Sushchansky constructed a finite family of infinite p-groups generated by two tableaux. Each such a tableau naturally defines an automorphism of a rooted tree and can be represented by a finite initial automaton. In paper [9] we describe these automata and study Sushchansky groups and their actions on rooted trees by means of this well-developed language.

The associated action of any Sushchansky group on a rooted tree is not level-transitive and we describe its orbit tree and show that there exists a faithful level-transitive action given by finite initial automata. Unlike the Grigorchuk group, Sushchansky groups are not self-similar. We consider a self-similar closure and prove that it is weakly regular branch, neither torsion nor torsion free and is generated by bounded automata, yielding contracting property and amenability. The question if the self-similar closure of any of Sushchansky groups is branch (or regular branch) is still open.

Our main result in the paper [9] is the following theorem, which is a contribution to the Milnor question on the existence of such groups, which was solved in [14] by R.I. Grigorchuk.

Theorem 8. *All Sushchansky groups have intermediate growth.*

Also we give an upper bound on the period growth function.

7 Automata generating free products of groups of order 2

All tree automorphisms defined by states of finite invertible automata over a fixed alphabet form a group of automatic transformations over this alphabet. The structure of this large group is yet to be understood. An interesting question is the embedding of known groups into this group. For example, Brunner and Sidki proved in [10] that $GL_n(\mathbb{Z})$ can be generated by finite automata over the alphabet with 2^n letters.

The first embedding of free products of groups C_2 of order 2 into the group of automatic transformations over the 2-letter alphabet was constructed by Olijnyk [22], but the group was not generated by all the states of corresponding automaton. The first self-similar example is the 3-state Bellaterra automaton $\mathcal{B}_3$ over 2-letter alphabet. This automaton has a direct connection to the famous 3-state Aleshin automaton generating the free group of rank 3 constructed by Aleshin in 1983. It took more than 20 years before the proof of this fact appeared in in [34]. Even though Aleshin did not prove that this automaton generates F_3, miraculously among 190 different non-symmetric 3-state automata over 2-letter alphabet, Aleshin automaton is the only one generating a free group as we show in [8].

It was proved [7, 21] that the Bellaterra automaton generates the group isomorphic to the free product of 3 copies of C_2. This automaton gives rise to a family of automata in which all states define involutive transformations. Namely, we modify the automaton $\mathcal{B}_3$ by inserting new states. More precisely, each automaton in the family is defined by wreath recursion

$$\begin{array}{l} a = (c, b), \quad b = (b, c), \quad c = (q_1, q_1)\sigma_0, \\ q_i = (q_{i+1}, q_{i+1})\sigma_i, \ i = 1, \ldots, n-4, \\ q_{n-3} = (a, a)\sigma_{n-3}, \end{array} \tag{1}$$

where $\sigma_i \in \mathrm{Sym}(\{0, 1\})$ is chosen arbitrarily.

Conjecturally, each nontrivial automaton in the family generates the free product of C_2. The first result supporting this conjecture was obtained by M. Vorobets and Y. Vorobets [35]. It was shown that if the number of states is odd and $\sigma_i = (12)$ for all i, then the conjecture holds. In the subsequent paper by the same authors and B. Steinberg [30], the conjecture was proved for the automata with even number of states and the additional condition that the number of nontrivial σ_i is odd.

In [27] (joint with Y. Vorobets) we prove

Theorem 9. *Each n-state automaton from the family (1) with $n \geq 4$ satisfying $\sigma_0 = (12)$ and $\sigma_{n-3} = (12)$ generates the free product of n copies of C_2.*

This covers the series constructed in [35] except the case $n = 3$, and overlaps with a family constructed in [30].

8 Software development

Groups and semigroups generated by automata are particularly nice from the computational point of view. Major algorithmic problems are unsolved so far in the general case but have solutions in certain special cases. Certain problems have several solutions of different complexity depending on the type of the group. For example, the general algorithm solving the word problem has exponential complexity, but there is a polynomial time algorithm for contracting groups, whose complexity bound I obtained in [28] in terms of the size of the nucleus of the group. The computations related to these groups are often cumbersome to be performed by hand and the computers may be very helpful here.

There was a strong need in the implementation of the algorithms related to automata groups and semigroups in some computer algebra system. The package `AutomGrp` [20] for `GAP`-system was developed by myself and Yevgen Muntyan to satisfy this need. The package was successfully used in the project of the classification of groups generated by 3-state automata over 2-letter alphabet [7], as well as by many other authors. The package is developing: currently I am working on incorporation of algorithms to determine finiteness of certain automaton groups described in recent paper [1]. I consider the support and developing of this package, as well as producing new pieces of software as an important part of my future research. I am happy to say that many people actually use the package, ask questions and make requests.

9 Classification of groups generated by automata

Similar to other classes of groups, the question of classification of groups generated by automata naturally arises. The first step in this direction was completed in [15], where it was proven that there are 6 nonisomorphic groups generated by 2-state automata over 2-letter alphabet.

I am involved in the project of classification of groups generated by 3-state automata over 2-letter alphabet. The results of our work were published in [7, 6, 5]. The situation here is much more complicated than in the case of 2-state automata. We have shown that there are no more then 115 nonisomorphic groups in the class. Substantial information about these groups was obtained: all finite groups and all abelian groups were detected, it was proved that there are no infinite torsion groups, and there is only one nonabelian free group is in the class, etc.

A natural continuation of this project would be the attempt to analyze the class of all groups generated by 4-state automata over 2-letter alphabet, or 2-state automata over 3-letter alphabet.

10 Future plans

Currently I have several ongoing projects. One (joint with R. Grigorchuk) is related to the classification of groups generated by 3-state automata over 2-letter alphabet that act essentially freely on the boundary of the tree. Another one (joint with M. Clay and L. Sabalka) deals with extension of the subsurface projection in mapping class group world to the submanifold projection in $Out(F_n)$ world. A new project (with A. Miasnikov) is related to graphs of intermediate growth whose incidence relation is accepted by finite automaton and their connection to Cayley automatic groups.

There are several unanswered questions listed in the previous sections which I am planning to address. In particular, it sounds very reasonable to generalize Theorem 6 to more general settings and to find condition on the action of the group that guarantees that this action can be restored from the Schreier graph of one orbit. Further, the technique used to construct Schreier graphs in [26] can be used for other groups acting on the Cantor set.

Another appealing to me question is to attack the algorithmic problems for self-similar groups, such as word problem, conjugacy problem, finding the order of an element, etc. Many partial solutions are known for special classes of groups, but it is not clear yet whether most of these problems are decidable or not in general. On the other hand, quite recently Z. Šunić and E. Ventura in [31] has shown that the conjugacy problem for automaton groups is generally undecidable. I plan to work in this direction because any work here will likely either bring some new algorithms, or may end up with an unsolvability result similar to the one mentioned above.

I will continue to develop the software for computations in group theory (undergraduate and graduate students can be incorporated in such projects).

Finally, I am interested in finding applications of groups generated by automata in noncommutative cryptography and coding theory. There is certain potential and not much has been done in this direction yet (see, for example, [13, 23]). And, of course, I plan to widen my research interests and will be open for collaboration with mathematicians in other fields of algebra and mathematics in general.

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