Exam I, Topology

October 5th, 2007

Please write your answers clearly and legibly. Unjustified answers will receive no credit. **There is a problem on the back. Remember to read the questions carefully before answering them. You must solve 5 of the 6 questions**. If you solve all 6, indicate which 5 you want graded. If you do all six, whichever one you don't want graded will be counted as some possible extra credit.

1. Recall that an ordered set A is called well-ordered if every non-empty subset of A has a smallest element. Prove that any well-ordered set A has the *least upper bound* property.

2. Let U_1 and U_2 be subsets of X, and let V_1 and V_2 be subsets of Y.

   (i) Prove that $(U_1 \times V_1) \bigcap (U_2 \times V_2) = (U_1 \bigcap U_2) \times (V_1 \bigcap V_2)$.

   (ii) Use the example where $U_1 = (0, 2) \subset \mathbb{R}$, $U_2 = (2, 4) \subset \mathbb{R}$, $V_1 = (-4, -2) \subset \mathbb{R}$, and $V_2 = (5, 8) \subset \mathbb{R}$ to show $(U_1 \times V_1) \bigcup (U_2 \times V_2) \neq (U_1 \bigcup U_2) \times (V_1 \bigcup V_2)$.

3. Let $\mathbb{R}$ be the set of real numbers. The standard topology $\mathcal{T}$ on $\mathbb{R}$ is the topology generated by the basis
$$\mathcal{B} = \{\, (a, b) \subset \mathbb{R} \,|\, a < b \,\}$$
of all open intervals in $\mathbb{R}$. The half-open (lower limit) topology $\mathcal{T}_\ell$ on $\mathbb{R}$ is the topology generated by the basis
$$\mathcal{B}_\ell = \{\, [a, b) \subset \mathbb{R} \,|\, a < b \,\}$$
of all half-open intervals closed on the left in $\mathbb{R}$. We can define a new topology on $\mathbb{R}$ called the **upper-limit** topology $\mathcal{T}_u$ as the topology generated by the basis
$$\mathcal{B}_u = \{\, (a, b] \subset \mathbb{R} \,|\, a < b \,\}$$
of all half-open intervals closed on the right in $\mathbb{R}$.

   (i) Show that $\mathcal{B}_u$ is a basis for a topology.

   (ii) Prove that the upper limit topology on $\mathbb{R}$ is strictly finer than the standard topology on $\mathbb{R}$ (i.e., prove that $\mathcal{T} \subsetneq \mathcal{T}_u$).

   (iii) (Extra Credit) Are the upper limit topology $\mathcal{T}_u$ on $\mathbb{R}$ and the lower limit topology $\mathcal{T}_\ell$ on $\mathbb{R}$ comparable? Explain. If so, which is finer?

4. Let $\mathbb{R}$ be the set of real numbers with the standard topology, let $\mathbb{Q}$ be the set of rational numbers, and let $\mathbb{Z}$ be the set of integers. We saw in class that the subspace topology on $\mathbb{Z}$ is the discrete topology. Is the subspace topology on $\mathbb{Q}$ the discrete topology? Give some reasoning for your answer.

5. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on a set X.

   (i) Does the collection
$$\mathcal{T}_1 \bigcap \mathcal{T}_2 = \{\, U \,|\, U \in \mathcal{T}_1 \text{ and } U \in \mathcal{T}_2 \,\}$$
   define a topology on X?

(ii) Does the collection

$$\mathcal{T}_1 \bigcup \mathcal{T}_2 = \{\, U \,|\, U \in \mathcal{T}_1 \text{ or } U \in \mathcal{T}_2 \,\}$$

define a topology on X?

6. For any set X, we can define the *finite complement topology* on X in the following way: A subset U of X is open if $X - U$ is either finite or all of X. We saw this in class for the case $X = \mathbb{R}$ the set of real numbers, however in general, this will define a topology on any set X.

   - Show that if $X = \{x_1, x_2, \ldots, x_n\}$ is a finite set, then the finite complement topology on X is the same as the discrete topology on X.