Exam 2, Topology

October 30th, 2007

1. Let X and Y be topological spaces. Prove that a bijective function $f : X \to Y$ is open if and only if it is closed. Prove that this is not true for a general function.

2. Show that any open interval (a, b) in $\mathbb{R}$ is homeomorphic to $(0, 1)$, and that any closed interval $[a, b]$ in $\mathbb{R}$ is homeomorphic to $[0, 1]$.

3. Let X and Y be topological spaces, and let $f : X \to Y$ be a function between X and Y. What can you say about whether or not f is continuous in the following cases? Can you say if f is open? Can you say if f is closed?

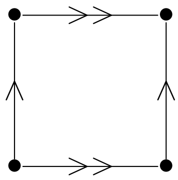
   (i) X has the discrete topology and Y has any topology.

   (ii) X has any topology and Y has the discrete topology.

   (iii) X has the indiscrete topology and Y has any topology.

   (iv) X has any topology and Y has the indiscrete topology.

4. Let T be the 2-dimensional torus. Let $\mathcal{T}_1$ be the the product topology on $T = S^1 \times S^1$ and let $\mathcal{T}_2$ be the quotient topology on T coming from seeing T as the identification space $T = (I \times I)/\sim$, pictured below, where $I = [0, 1]$ is the closed interval in $\mathbb{R}$.

   The identification in the picture is given by $(0, y) \sim (1, y)$ and $(x, 0) \sim (x, 1)$ for all $x, y \in I$. No other identifications are made. Show that $\mathcal{T}_1$ and $\mathcal{T}_2$ are the same topology.

5. Let $X = \mathbb{R} \times \{0, 1\}$. Define an equivalence relation on X by $(x, 0) \sim (x, 1)$ if $x \neq 0$. The identification space $\widehat{X}$ is called **the line with two origins**. Give $\widehat{X}$ the quotient topology.

   (i) Show that $\widehat{X}$ is not Hausdorff.

   (ii) Give a sequence in $\widehat{X}$ that converges to both origins, i.e. to $(0, 0)$ and to $(0, 1)$.

   (iii) Give a sequence in $\widehat{X}$ that converges to just $(0, 0)$.

6. Let X be the identification space obtained from $\mathbb{R}$ under the relation $x \sim y$ iff $x - y \in \mathbb{Q}$. Give X the quotient topology. Show that the only open sets in X are X and ϕ which means X has the indiscrete topology.