Exam 3, Topology

November 20, 2007

Justify your answers.

1. Prove that $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}$ for any $n > 1$. [Hint: Show that there aren't any injective continuous functions $f : \mathbb{R}^n \to \mathbb{R}$.]

2. Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on the set X. Show that if X is compact and Hausdorff under both topologies, then either $\mathcal{T} = \mathcal{T}'$ or $\mathcal{T}$ and $\mathcal{T}'$ are not comparable.

3. Show that every compact subspace of a metric space is closed and bounded in the metric. Find a metric space in which not every closed bounded subspace is compact.

4. Show that if $f : X \to Y$ is continuous, where X is compact and Y is Hausdorff, then f is a closed map.

5. Let $p : X \to Y$ be a closed, continuous, surjective function such that $p^{-1}(\{y\})$ is compact for each $y \in Y$. Show that if Y is compact, then X is compact. [Hint: If U is an open set containing $p^{-1}(\{y\})$, there is a neighborhood W of y such that $p^{-1}(W)$ is contained in U.]

6. Let $\mathbb{R}_{fc}$ denote the real numbers with the finite complement topology, and let $\mathbb{R}_\ell$ denote the real numbers with the lower limit (half open) topology.

   (a) Is $\mathbb{R}_{fc}$ compact?

   (b) If $A \subset \mathbb{R}_{fc}$ is a subspace, is A compact?

   (c) Is $\mathbb{R}_\ell$ compact?

   (d) If $[a, b] \subset \mathbb{R}_\ell$ has the subspace topology, is $[a, b]$ compact?

7. Let X be a space with the discrete topology. Show that X is compact if and only if X is finite as a set (i.e. X has a finite number of points).