

Final Exam, Topology

December 12, 2007

You have some choices here on the final. **You must do 6 of the 8 problems listed with the following conditions:**

- You must do 1 of the first 2 problems (#1-3).
- You must do problems #3 and #4.
- You must do 3 of the last 4 problems (#5-8).

Justify your answers.

1. A group G with operation \cdot is called a topological group if G has a topology in which one point sets are closed, and the functions

$$\mu : G \times G \rightarrow G \quad \text{with } \mu(x, y) = x \cdot y \quad \text{and } \iota : G \rightarrow G \quad \text{with } \iota(x) = x^{-1}$$

are both continuous. In this case, $G \times G$ is given the product topology.

- (i) Show that the real numbers \mathbb{R} under addition form a topological group.
 - (ii) Show that the circle S^1 under multiplication is a topological group. Here S^1 is taken to be the space of all complex numbers z for which $|z| = 1$ and multiplication is the multiplication of complex numbers. Recall that if $z = x + y\sqrt{-1}$, then $|z| = \sqrt{x^2 + y^2}$.
 - (iii) Show that the *general linear group* $GL(2, \mathbb{R})$ under the operation of matrix multiplication is a topological group. Recall that $GL(n, \mathbb{R})$ is the set of all nonsingular n by n matrices with real number entries, and a matrix A is nonsingular if the determinant of A is not 0. The topology on $GL(n, \mathbb{R})$ is the subspace topology coming from thinking of $GL(n, \mathbb{R})$ as a subspace of \mathbb{R}^{n^2} in the obvious way.
2. Let x_0 and x_1 be points of the path-connected space X . Show that $\pi_1(X, x_0)$ is abelian if and only if for every pair α and β of paths from x_0 to x_1 , we have $\hat{\alpha} = \hat{\beta}$. Recall that an abelian group G with operation \cdot is a group which satisfies $x \cdot y = y \cdot x$ for all $x, y \in G$.
 3. Let G be a topological group with operation \cdot and identity element x_0 . Let

$$\Omega(G, x_0) = \{ f \mid f : I \rightarrow G, f(0) = f(1) = x_0 \}$$

denote the set of all loops in G based at x_0 . If $f, g \in \Omega(G, x_0)$, let us define a loop $f \otimes g$ by the rule

$$(f \otimes g)(s) = f(s) \cdot g(s).$$

- (i) Show that the operation \otimes makes the set $\Omega(G, x_0)$ into a group.
- (ii) Show that the operation \otimes induces a group operation \otimes on $\pi_1(G, x_0)$.
- (iii) Show that the two group operations \otimes and $*$ on $\pi_1(G, x_0)$ are the same. In other words, show $[f] \otimes [g] = [f] * [g]$. [Hint: Compute $(f * e_{x_0}) \otimes (e_{x_0} * g)$ and remember $f * e_{x_0} \simeq f$.]
- (iv) Show that $\pi_1(G, x_0)$ is an abelian group. In other words, show $[f] * [g] = [g] * [f]$. [Hint: Compute $(f * e_{x_0}) \otimes (e_{x_0} * g)$ and $(e_{x_0} * f) \otimes (g * e_{x_0})$. Use those computations to get the result.]

4. A continuous function $f : X \rightarrow Y$ between topological spaces is called a *homotopy equivalence* with *homotopy inverse* $g : Y \rightarrow X$ if $f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$. Here $id_X : X \rightarrow X$ is the identity function on X , and id_Y is the identity function on Y . For any space Z , let $[X, Z]$ be the set of homotopy classes of continuous functions from X to Z . For a continuous map $h : X \rightarrow Z$, let $[h]$ denote the homotopy class of h in $[X, Z]$.

- (i) Given continuous maps $f : X \rightarrow Y$ and $g : Y \rightarrow Z$, we can define the composition $g \circ f : X \rightarrow Z$. Explain why this gives a well-defined operation at the level of homotopy classes,

$$[g] \circ [f] = [g \circ f].$$

[Hint: You have already proven this fact in a previous homework.]

- (ii) Using part (i), show that for continuous maps $f : X \rightarrow Y$, $g : Y \rightarrow Z$, and $h : Z \rightarrow W$ we have

$$[id_Y] \circ [f] = [f] = [f] \circ [id_X] \quad \text{and} \quad ([h] \circ [g]) \circ [f] = [h] \circ ([g] \circ [f]).$$

For the last two parts, let $f : X \rightarrow Y$ be a homotopy equivalence with homotopy inverse $g : Y \rightarrow X$.

- (iii) Show that for any topological space Z , there is a bijection between the set $[Z, Y]$ and the set $[Z, X]$. [Hint: Define a function $\phi : [Z, Y] \rightarrow [Z, X]$ by $\phi([h]) = [g] \circ [h]$.]
- (iv) Show that for any topological space W , there is a bijection between the set $[X, W]$ and the set $[Y, W]$.

5. Let $K = \{\frac{1}{n} | n \in \mathbb{Z}_+\}$, and let \mathcal{T}_K be the topology of \mathbb{R} in which the basic open sets are open intervals (a, b) and sets of the form $(a, b) - K$. This topology appeared in Homework 3, problem 6. We will let \mathbb{R}_K denote the real numbers with the topology \mathcal{T}_K .

(i) Is \mathbb{R}_K a Hausdorff space?

(ii) Is \mathbb{R}_K a compact space?

(iii) Is \mathbb{R}_K a connected space?

6. Let $f, g : X \rightarrow Y$ be continuous maps between topological spaces, and let Y be Hausdorff. Show that $\{x \in X | f(x) = g(x)\}$ is a closed subset of X .

7. Let $p : X \rightarrow Y$ be a closed continuous surjective map such that $p^{-1}(\{y\})$ is compact for every $y \in Y$. Such a map p is sometimes called a perfect map. Show that if X is Hausdorff, then so is Y .

8. Let $f : X \rightarrow X$ be continuous.

(i) Show that if $X = [0, 1]$, there is a point x such that $f(x) = x$. The point x is called a *fixed point* of f .

(ii) What happens if $X = [0, 1)$ or $X = (0, 1)$? Does your argument still work?