

Homework 3

Math 518

November 17, 2006

1. Derive the **Wang long exact sequence** for a fibration $F \hookrightarrow E \rightarrow B$ with B a simply connected homology n -sphere and F connected,

$$\cdots \longrightarrow H^k(E; R) \xrightarrow{i^*} H^k(F; R) \xrightarrow{\theta} H^{k-n+1}(F; R) \xrightarrow{j^*} H^{k+1}(E; R) \longrightarrow \cdots$$

Show further that θ is a derivation when n is even, and when n is odd, θ is an antiderivation, i.e. $\theta(x \cup y) = \theta(x) \cup y + x \cup \theta(y)$.

2. Establish the Serre exact sequence in the case of homology.
3. The symplectic groups $Sp(n)$ are the analogues of the special orthogonal or unitary groups over the quaternions. There are analogous fibrations $Sp(n-1) \hookrightarrow Sp(n) \rightarrow S^{4n-3}$. Given $Sp(1) \cong S^3$, compute $H^*(Sp(n); \mathbb{Z})$ for all n .
4. Continue the argument that we made in class in order to show

$$H^i(K(\mathbb{Z}, 3); \mathbb{Z}) = \begin{cases} \mathbb{Z}_3 y & i = 8 \\ \mathbb{Z}_2 x^3 & i = 9 \\ \mathbb{Z}_2 z & i = 10 \\ \mathbb{Z}_3 xy & i = 11 \end{cases} .$$

5. The quaternionic Stiefel manifolds $V_k(\mathbb{H}^n)$ are defined similarly to the real and complex Stiefel manifolds. Compute $H^*(V_k(\mathbb{H}^n); \mathbb{Z})$.
6. Let $S^{n-1} \hookrightarrow V_2(\mathbb{R}^{n+1}) \rightarrow S^n$ be the fibration sending the orthonormal 2 frame (v_1, v_2) to v_1 . Show that this fibration has no section if n is even.

- Use the given fibration to show that $\pi_i(S^{2m})$ is finite for $0 < i < 4m$ except when $i = 2m$ or $i = 4m - 1$. In particular, show that $\pi_{4m-1}(S^{2m}) \cong \mathbb{Z} \oplus$ finite. (Hint: Compute $H_*(V_2(\mathbb{R}^{n+1}); \mathbb{Q})$ and apply the rational Hurewicz theorem. Then look at the long exact sequence of homotopy groups for the fibration.)
- Modify the argument we used to show $\pi_i(S^{2n-1})$ is finite for $i > 2n - 1$ to show that in fact $\pi_i(S^{2m})$ is finite for $i \geq 4m$ with the following steps.
 - * Form the fibration $F \hookrightarrow S^{2m} \rightarrow K(\mathbb{Z}, 2m)$ and look at the Leray-Serre spectral sequence in cohomology with \mathbb{Q} -coefficients.
 - * Calculate $H^*(F; \mathbb{Q})$.
 - * Assume we have a space F_{4m-2} with $\pi_i(F_{4m-2}) \cong \pi_i(F)$ for $i \leq 4m - 2$ and $\pi_i(F_{4m-2}) = 0$ for $i > 4m - 2$, and a map $p : F \rightarrow F_{2m-2}$ which realizes the isomorphism on lower homotopy groups. After replacing the map by a fibration with fibre F' , show $\pi_i(F') \cong \pi_i(S^{2m})$ for $i \geq 4m - 1$.
 - * Show $H^*(F'; \mathbb{Q}) \cong H^*(F; \mathbb{Q})$ using the fact that $\pi_i(F_{2m-2})$ are finite for all i (what does this say about $H^*(F_{4m-2}; \mathbb{Q})$?) and the spectral sequence for the fibration $F' \hookrightarrow F \rightarrow F_{4m-2}$.
 - * Since $\pi_{4m-1}(F') = \mathbb{Z} \oplus$ finite, there is a map $f : F' \rightarrow K(\mathbb{Z}, 4m - 1)$ which induces an isomorphism on the \mathbb{Z} part of π_{4m-1} . After replacing f by a fibration, deduce that $\pi_i(F')$ is finite for $i \geq 4m$ using the same argument we used to show $\pi_i(S^{2n-1})$ is finite for $i > 2n - 1$.