Quiz 5, Calc 2, Section 3

March 5th, 2007

1. Evaluate the following integrals

   (a) $\displaystyle\int_1^4 \frac{1}{(x-2)^{2/5}}\,dx$

   (b) $\displaystyle\int_2^{+\infty} \frac{1}{x\ln x}\,dx$

Solution: (a) The first thing to notice is that this is an improper integral since $f(x) = \dfrac{1}{(x-2)^{2/5}}$ is not continuous at $x = 2$. So we have to split it up into

$$\int_1^4 \frac{1}{(x-2)^{2/5}}\,dx = \int_1^2 \frac{1}{(x-2)^{2/5}}\,dx + \int_2^4 \frac{1}{(x-2)^{2/5}}\,dx.$$

Now we need to do each integral seperately. So let's do the first integral. We need to rewrite it as a limit, to get

$$\begin{aligned}
\int_1^2 \frac{1}{(x-2)^{2/5}}\,dx &= \lim_{t\to 2^-} \int_1^t \frac{1}{(x-2)^{2/5}}\,dx \\
&= \lim_{t\to 2^-} \frac{5}{3}(x-2)^{3/5}\Big|_1^t \\
&= \lim_{t\to 2^-} \left(\frac{5}{3}(t-2)^{3/5} - \frac{5}{3}(1-2)^{3/5}\right) \\
&= -\frac{5}{3}(-1)^{3/5} = \frac{5}{3}.
\end{aligned}$$

So the first integral converges. For the second integral we have

$$\begin{aligned}
\int_2^4 \frac{1}{(x-2)^{2/5}}\,dx &= \lim_{t\to 2^+} \int_t^4 \frac{1}{(x-2)^{2/5}}\,dx \\
&= \lim_{t\to 2^+} \frac{5}{3}(x-2)^{3/5}\Big|_t^4 \\
&= \lim_{t\to 2^+} \left(\frac{5}{3}(4-2)^{3/5} - \frac{5}{3}(t-2)^{3/5}\right) \\
&= \frac{5}{3}(2)^{3/5}.
\end{aligned}$$

So the second integral converges. That means that our original integral converges and

$$\begin{aligned}\int_1^4 \frac{1}{(x-2)^{2/5}}\,dx &= \int_1^2 \frac{1}{(x-2)^{2/5}}\,dx + \int_2^4 \frac{1}{(x-2)^{2/5}}\,dx \\ &= \frac{5}{3} + \frac{5}{3}(2^{3/5}).\end{aligned}$$

(b) This is an improper integral since we have $+\infty$ as one of the endpoints of integration. So we rewrite the integral as a limit

$$\int_2^{+\infty} \frac{1}{x\ln x}\,dx = \lim_{t\to+\infty} \int_2^t \frac{1}{x\ln x}\,dx.$$

Now we need to evaluate the integral. To do this, I'm going to do the integral seperately so that I can plug it into the limit. If we let $u = \ln x$, then $du = \frac{1}{x}\,dx$. This gives

$$\begin{aligned}\int \frac{1}{x\ln x}\,dx &= \int \frac{1}{u}\,du \\ &= \ln|u| + C \\ &= \ln|\ln x| + C.\end{aligned}$$

Plugging this result into our limit, we have

$$\begin{aligned}\int_2^{+\infty} \frac{1}{x\ln x}\,dx &= \lim_{t\to+\infty} \int_2^t \frac{1}{x\ln x}\,dx \\ &= \lim_{t\to+\infty} (\ln|\ln x|)\big|_2^t \\ &= \lim_{t\to+\infty} (\ln|\ln t| - \ln|\ln 2|) \\ &= +\infty.\end{aligned}$$

Therefore this integral diverges.

2. Determine if the following series converges. If it converges, find the sum.

$$3 - \frac{9}{5} + \frac{27}{25} - \frac{81}{125} + \frac{243}{625} - \cdots$$

Solution: This series is a geometric series, which we can see since the numbers we are adding have a geometric pattern. The pattern is that we multiply each term in the sum by $-3/5$ to get the next term in the sum. So this is a geometric series with $r = -3/5$. Since $|r| < 1$, the series will converge.

To figure out what it converges to, we need to use the formula we have for geometric series. The formula is that if a geometric series converges, then it converges to $\frac{a}{1-r}$. Here a is the first term in the series, and r is the common ratio. For the series we're looking at $a = 3$ and (as we already saw) $r = -3/5$. So our series converges to

$$\frac{3}{1 - \left(-\frac{3}{5}\right)} = \frac{3}{1 + \frac{3}{5}} = \frac{3}{\frac{8}{5}} = \frac{15}{8}.$$