

## Quiz 4, Calc 2, Section 9

March 5th, 2007

1. Evaluate the following integrals

$$(a) \int_1^4 \frac{1}{(x-2)^{2/3}} dx$$

$$(b) \int_2^{+\infty} \frac{1}{x \ln x} dx$$

*Solution:* (a) The first thing to notice is that this is an improper integral since  $f(x) = \frac{1}{(x-2)^{2/3}}$  is not continuous at  $x = 2$ . So we have to split it up into

$$\int_1^4 \frac{1}{(x-2)^{2/3}} dx = \int_1^2 \frac{1}{(x-2)^{2/3}} dx + \int_2^4 \frac{1}{(x-2)^{2/3}} dx.$$

Now we need to do each integral separately. So let's do the first integral. We need to rewrite it as a limit, to get

$$\begin{aligned} \int_1^2 \frac{1}{(x-2)^{2/3}} dx &= \lim_{t \rightarrow 2^-} \int_1^t \frac{1}{(x-2)^{2/3}} dx \\ &= \lim_{t \rightarrow 2^-} 3(x-2)^{1/3} \Big|_1^t \\ &= \lim_{t \rightarrow 2^-} \left( 3(t-2)^{1/3} - 3(1-2)^{1/3} \right) \\ &= -3(-1)^{3/5} = 3. \end{aligned}$$

So the first integral converges. For the second integral we have

$$\begin{aligned} \int_2^4 \frac{1}{(x-2)^{2/3}} dx &= \lim_{t \rightarrow 2^+} \int_t^4 \frac{1}{(x-2)^{2/3}} dx \\ &= \lim_{t \rightarrow 2^+} 3(x-2)^{1/3} \Big|_t^4 \\ &= \lim_{t \rightarrow 2^+} \left( 3(4-2)^{1/3} - 3(t-2)^{1/3} \right) \\ &= 3(2)^{1/3}. \end{aligned}$$

So the second integral converges. That means that our original integral converges and

$$\begin{aligned} \int_1^4 \frac{1}{(x-2)^{2/3}} dx &= \int_1^2 \frac{1}{(x-2)^{2/3}} dx + \int_2^4 \frac{1}{(x-2)^{2/3}} dx \\ &= 3 + 3(2^{1/3}). \end{aligned}$$

(b) This is an improper integral since we have  $+\infty$  as one of the endpoints of integration. So we rewrite the integral as a limit

$$\int_2^{+\infty} \frac{1}{x \ln x} dx = \lim_{t \rightarrow +\infty} \int_2^t \frac{1}{x \ln x} dx.$$

Now we need to evaluate the integral. To do this, I'm going to do the integral separately so that I can plug it into the limit. If we let  $u = \ln x$ , then  $du = \frac{1}{x} dx$ . This gives

$$\begin{aligned} \int \frac{1}{x \ln x} dx &= \int \frac{1}{u} du \\ &= \ln |u| + C \\ &= \ln |\ln x| + C. \end{aligned}$$

Plugging this result into our limit, we have

$$\begin{aligned} \int_2^{+\infty} \frac{1}{x \ln x} dx &= \lim_{t \rightarrow +\infty} \int_2^t \frac{1}{x \ln x} dx \\ &= \lim_{t \rightarrow +\infty} (\ln |\ln x|)|_2^t \\ &= \lim_{t \rightarrow +\infty} (\ln |\ln t| - \ln |\ln 2|) \\ &= +\infty. \end{aligned}$$

Therefore this integral diverges.

2. Determine if the following series converges. If it converges, find the sum.

$$3 - \frac{9}{5} + \frac{27}{25} - \frac{81}{125} + \frac{243}{625} - \dots$$

*Solution:* This series is a geometric series, which we can see since the numbers we are adding have a geometric pattern. The pattern is that we multiply each term in the sum by  $-3/5$  to get the next term in the sum. So this is a geometric series with  $r = -3/5$ . Since  $|r| < 1$ , the series will converge.

To figure out what it converges to, we need to use the formula we have for geometric series. The formula is that if a geometric series converges, then it converges to  $\frac{a}{1-r}$ . Here  $a$  is the first term in the series, and  $r$  is the common ratio. For the series we're looking at  $a = 3$  and (as we already saw)  $r = -3/5$ . So our series converges to

$$\frac{3}{1 - (-\frac{3}{5})} = \frac{3}{1 + \frac{3}{5}} = \frac{3}{\frac{8}{5}} = \frac{15}{8}.$$