Some notes on Infinite Series, **Calc 2**
March 12th, 2007

1 Geometric and p-series

An infinite series is a geometric series if it of the form

$$\sum_{n=1}^{+\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + ar^4 + \cdots$$

The series has a pattern that each term in the sum is r times the previous term, where r is called the common ratio. For example the series

$$6 - \frac{18}{4} + \frac{54}{16} - \frac{162}{64} + \frac{486}{256} - \cdots$$

is a geometric series with common ratio $-3/4$.

Whether or not a geometric series converges depends entirely on the common ratio r. The series will converge if $|r| < 1$, and it will diverge if $|r| > 1$. So we can always tell if a geometric series will converge. Not only that, but if a geometric series converges, there is a formula that tells us what the geometric series will converge to. If a geometric series converges, then it converges to $\frac{a}{1-r}$. Here a is the first term of the series, and r is the common ratio.

For example if we again consider the series

$$6 - \frac{18}{4} + \frac{54}{16} - \frac{162}{64} + \frac{486}{256} - \cdots$$

we already saw that this is a geometric series with $r = -3/4$. Well, since $|r| = |-3/4| < 1$, the series converges. Using the formula, we see

$$6 - \frac{18}{4} + \frac{54}{16} - \frac{162}{64} + \frac{486}{256} - \cdots = \frac{6}{1 - \left(-\frac{3}{4}\right)} = \frac{24}{7}.$$

It is usually pretty easy to tell if a series is a geometric series. If when you look at the series, the formula for the term only has n appearing in an exponent, then it might be a geometric series. If you think a series might be geometric, you can write out the first four or five terms to see if it you can find a common ratio.

For example.

$$\sum_{n=2}^{\infty} \frac{3^{n+3}}{7^{2n}}$$

looks like it might be a geometric series, and if we write out a few terms

$$\sum_{n=2}^{\infty} \frac{3^{n+3}}{7^{2n}} = \frac{3^5}{7^4} + \frac{3^6}{7^6} + \frac{3^7}{7^8} + \frac{3^8}{7^{10}} + \frac{3^9}{7^{12}} + \cdots$$

we can see that it is geometric with a common ratio $r = 3/(7^2)$. The series

$$\sum_{n=1}^{\infty} \frac{2^n}{5 + 3^n}$$

is not geometric. Again, if you write out a few terms you can see that. The probelm is that there is no way to split up the term because of the addition in the bottom of the fraction.

Another important type of series is p-series. These will be especially important when we use the Comparison and Limit Comparison tests. A p-series is a series of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^p}.$$

A p-series will converge if $p > 1$, and it will diverge if $p \leq 1$.

2 Tests that we have

Here is a listing of the tests that we can use to figure out if a series converges or diverges.

- **Test for Divergence:** If $\lim\limits_{n\to\infty} a_n \neq 0$, then $\sum a_n$ diverges. Remember if $\lim\limits_{n\to\infty} a_n = 0$, the test doesn't tell you anything.
- **The Integral Test:** Let $f(x)$ be a continuous, positive, decreasing function on $[1, +\infty)$ with $f(n) = a_n$ for all n. Then $\sum\limits_{n=1}^{\infty} a_n$ converges if and only if $\int_1^{\infty} f(x)\,dx$ converges. In other words the integral and the series do the same thing. Either both converge, or both diverge.
- **The Comparison Test:** Let $0 \leq a_n \leq b_n$ for all n. Then
  1. $\sum a_n$ converges if $\sum b_n$ converges.
  2. $\sum b_n$ diverges if $\sum a_n$ diverges.

  In other words, if the bigger series converges, the smaller one converges, and if the smaller series diverges, then the bigger one diverges.
- **The Limit Comparison Test:** Let $\sum a_n$ and $\sum b_n$ be positive series such that

  $$\lim_{n\to\infty} \frac{a_n}{b_n} = L$$

  exists. Then
  1. if $0 < L < +\infty$, then $\sum a_n$ converges if and only if $\sum b_n$ converges. In other words the two series do the same thing.
  2. if $L = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.

(3) if $L = +\infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

- **The Alternating Series Test:** Let $\sum(-1)^{n+1}a_n$ be an alternating series ($a_n > 0$ for all n) with $a_{n+1} \leq a_n$ for all n and $\lim_{n\to\infty} a_n = 0$, then $\sum(-1)^{n+1}a_n$ converges.
- **The Ratio Test:** Let $\sum a_n$ be any series,
  (1) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = r < 1$, the series $\sum a_n$ converges absolutely.
  (2) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = r > 1$, the series $\sum a_n$ diverges.
  (3) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = r = 1$, the test fails and we get NO INFORMATION.
- **The Root Test:** Let $\sum a_n$ be any series,
  (1) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = r < 1$, the series $\sum a_n$ converges absolutely.
  (2) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = r > 1$, the series $\sum a_n$ diverges.
  (3) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = r = 1$, the test fails and we get NO INFORMATION.

You should also know the definition for absolute convergence and conditional convergence.

- The series $\sum a_n$ converges absolutely if $\sum |a_n|$ converges.
- The series $\sum a_n$ converges conditionally if $\sum a_n$ converges but $\sum |a_n|$ diverges.

Remember, the only series we will ever see that converge condiitonally are alternating series.

3 Strategies for testing a series

Here we should think about some strategies for using the tests that we have. It's not a good idea to just try all the test in order on a series until we find one that works. Instead, we want to have an idea of what test to use based on how the series looks.

Given a series $\sum a_n$,

1. When you first look at the series, try the Test for Divergence quickly. You don't have to write it down, and you can do it in your head easily for most series. If $\lim_{n\to\infty} a_n \neq 0$, then the series diverges and you are done. In this case you should write down the limit. If $\lim_{n\to\infty} a_n = 0$, then the Test for Divergence doesn't tell you anything, so you have to try something else.

2. If the series 'looks like' a p-series or a geometric series, then you could try to do the Limit Comparison Test or the Comparison Test. If a_n is a rational function of n or something involving roots of polynomials of n, then the Limit Comparison test with a p-series is the way to go.

3. If the series involves factorials or other products (including a constant raised to the n^{th} power), then the Ratio Test is a good possibility.

4. If a_n is of the form $(b_n)^n$, then the Root Test might be the best option.

5. If the series is an alternating series, then the Alternating Series test is an obvious way to go.

6. If $a_n = f(n)$, where $\int_1^\infty f(x)\,dx$ is an integral that you can do, then the Integral Test is a possibility as long as $f(x)$ satisfies the criteria (contiuous, positive, decreaasing).

7. If the series is a geometric series, then you only need to look at the common ratio r. If $|r| < 1$, the series will converge. If $|r| > 1$, the series will diverge. You might need to do aome algebra to get to the point that you recognize that it is a geometric series.

8. If the series is a p-series, then you only need to look at p. The series will converge if $p > 1$, and it will diverge if $p \leq 1$.

I will admit that the two tests I will most often use are the Limit Comparison Test and the Ratio Test. A lot of series can be dealt with using these two tests. However, you do need to know how to use all the tests we have.

When you are dealing with a series that has some negative terms, the first thing to do is to check if it converges absolutely. If it is an alternating series and it doesn't converge absolutely, then you should try the alternating series test.

4 Exercises

Decide if the foloowing series converge or diverge. If it converges say if it converges absolutely.

1. $\displaystyle\sum_{n=2}^{\infty} \frac{n^3 - 4n^2 + 1}{5n^7 + 15n^3 - 12}$

2. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln(n)}$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{(2n)!}$

4. $\sum_{n=5}^{\infty} \left(\frac{n+1}{n} \right)^n$

5. $\sum_{n=2}^{\infty} \frac{(-1)^{n+1} 5^{3n}}{7^{2n+1}}$

6. $\sum_{n=2}^{\infty} \frac{n+3}{3n^2 - 2}$

7. $\sum_{n=2}^{\infty} \frac{cos(n\pi)}{n^{1.01}}$