Homework 1, **Topology I**

Due September 5th, 2007

1. Prove the properties $A \bigcap (B \bigcup C) = (A \bigcap B) \bigcup (A \bigcap C)$ and $A \bigcup (B \bigcap C) = (A \bigcup B) \bigcap (A \bigcup C)$.

2. Let $\mathbb{R}$ be the set of real numbers, and let $\mathbb{Z}$ be the set of integers. For each of the following subsets of $\mathbb{R} \times \mathbb{R}$, determine whether it is equal to the cartesian product of two subsets of $\mathbb{R}$. Justify your answer.

   (a) $\{(x, y) \,|\, x \in \mathbb{Z}\}$.
   (b) $\{(x, y) \,|\, y > x\}$.
   (c) $\{(x, y) \,|\, x \notin \mathbb{Z} \text{ and } y \in \mathbb{Z}\}$.
   (d) $\{(x, y) \,|\, x^2 + y^2 < 4\}$.

3. Let $f : A \to B$ be a function between sets. Let $A_0 \subset A$ and $B_0 \subset B$.

   (a) Show that $A_0 \subset f^{-1}(f(A_0))$ and that equality holds if f is injective.
   (b) Show that $f(f^{-1}(B_0)) \subset B_0$ and that equality holds if f is surjective.

4. Let $f : A \to B$ be a function between sets, and let $A_i \subset A$ and $B_i \subset B$ for $i = 0, 1$. Show that f^{-1} preserves inclusions, unions, intersections, and differences of sets.

   (a) $B_0 \subset B \Rightarrow f^{-1}(B_0) \subset f^{-1}(B)$.
   (b) $f^{-1}(B_0 \bigcup B_1) = f^{-1}(B_0) \bigcup f^{-1}(B_1)$.
   (c) $f^{-1}(B_0 \bigcap B_1) = f^{-1}(B_0) \bigcap f^{-1}(B_1)$.
   (d) $f^{-1}(B_0 - B_1) = f^{-1}(B_0) - f^{-1}(B_1)$.

   Show that f only preserves inclusions and unions.

   (e) $A_0 \subset A_1 \Rightarrow f(A_0) \subset f(A_1)$.
   (f) $f(A_0 \bigcup A_1) = f(A_0) \bigcup f(A_1)$.
   (g) $f(A_0 \bigcap A_1) \subset f(A_0) \bigcap f(A_1)$; show that equality holds if f is injective.
   (h) $f(A_0 - A_1) \supset f(A_0) - f(A_1)$; show that equality holds if f is injective.

5. Define two points (x_0, y_0) and (x_1, y_1) in the plane $\mathbb{R}^2$ to be equivalent if $y_0 - x_0^2 = y_1 - x_1^2$. Check that this is an equivalence relation, and describe the equivalence classes.

6. Here is a "proof" that every relation which is both symmetric and transitive is also reflexive: "Since C is symmetric, aCb implies bCa. Since C is transitive, aCb and bCa together imply aCa as desired." Where is the flaw in this argument?

7. Prove the following
   Theorem: If an ordered set A has the least upper bound property, then it has the greatest lower bound property.

8. Assume that the real line has the least upper bound property.

   (a) Show that the sets
   $$[0, 1] = \{x \,|\, 0 \le x \le 1\}$$
   $$[0, 1) = \{x \,|\, 0 \le x < 1\}$$
   have the least upper bound property.
   (b) Does $[0, 1] \times [0, 1]$ with the dictionary ordering have the least upper bound property? What about $[0, 1] \times [0, 1)$? How about $[0, 1) \times [0, 1]$?

9. Show that if B is not finite and $B \subset A$, then A is not finite.

10. If A and B are finite sets, show that the set of all functions $f : A \to B$ is finite.