Homework 2, Topology I

Due Monday September 17th, 2007

1. Give the set $A = [0,1] \times [0,1)$ the dictionary ordering. Define $B = \{\, (x,y) \,|\, 0 \leq x \leq 0.5,\ 0 \leq y < 1 \,\}$. It is clear that $B \subset A$. Show that B has an upper bound in A, but that B does not have a least upper bound in A. This proves that A does not have the least upper bound property.

2. If $A \times B$ is a finite set, does it follow that A and B are finite? Justify your answer.

3. Let A be a finite simply ordered set with order relation $<$. Prove that A has a largest element. [Hint: Proceed by induction on the cardinality of A]

4. Show that $\mathbb{Q}$ is countably infinite.

5. A real number is said to be **algebraic** (over the rationals) if it satisfies some polynomial equation of poisitive degree

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$$

with rational coefficient a_j ($a_j \in \mathbb{Q}$ for $0 \leq j \leq n-1$). Assuming that each polynomial equation has only finitely many roots, show that the set of algebraic real numbers is countable.

6. A real number is said to be **transcendental** if it is not algebraic. Assuming that the reals are uncountable, show that the set of transcendental numbers is uncountable. (It is a little surprising that only two transcendental numbers are commonly known: e and π. Even proving that those two are transcendental is very non-trivial.)

7. For each of the following sets, determine whether or not it is countable. Justify your answer.

   (a) The set A of all functions $f : \{0,1\} \to \mathbb{Z}_+$.

   (b) The set B_n of all functions $f : \{1,2,\ldots,n\} \to \mathbb{Z}_+$.

   (c) The set C of all functions $f : \mathbb{Z}_+ \to \mathbb{Z}_+$.

   (d) The set D of all functions $f : \mathbb{Z}_+ \to \{0,1\}$.