Homework 3, **Topology I**

Due Monday October 1st, 2007

1. Consider the set $\mathbb{R}$ of real numbers. In class, we defined a topology $\mathcal{T}_{fc}$ on $\mathbb{R}$ called the finite complement topology. In this topology, a subset U in $\mathbb{R}$ is open if $\mathbb{R} - U$ is either finite or all of $\mathbb{R}$.

   (a) Let $\mathcal{B}$ be the collection of all open intervals (a, b) in $\mathbb{R}$. $\mathcal{B}$ is a basis for a topology on $\mathbb{R}$. Let $\mathcal{T}$ be the topology generated by $\mathcal{B}$. This is called the standard topology on $\mathbb{R}$. Are $\mathcal{T}_{fc}$ and $\mathcal{T}$ comparable? If so, which one is larger?

   (b) Let $\mathcal{B}'$ be the collection of all half-open intervals $[a, b)$ in $\mathbb{R}$. $\mathcal{B}'$ is a basis for a topology on $\mathbb{R}$. Let $\mathcal{T}_\ell$ be the topology generated by $\mathcal{B}'$. This is called the half-open topology on $\mathbb{R}$ (It is also called the lower limit topology on $\mathbb{R}$). Are $\mathcal{T}_{fc}$ and $\mathcal{T}_\ell$ comparable? If so, which one is larger?

2. Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on the set X with $\mathcal{T} \subsetneq \mathcal{T}'$. What can you say about the corresponding subspace topologies on the subset Y of X?

3. Consider the set $Y = [-1, 1]$ as a subspace of $\mathbb{R}$ with the standard topology. Which of the following sets are open in Y? Which are open in $\mathbb{R}$?

   (i) $A = \{ x \mid \frac{1}{2} < |x| < 1 \}$

   (ii) $B = \{ x \mid \frac{1}{2} < |x| \leq 1 \}$

   (iii) $C = \{ x \mid \frac{1}{2} \leq |x| < 1 \}$

   (iv) $D = \{ x \mid \frac{1}{2} \leq |x| \leq 1 \}$

   (v) $E = \{ x \mid 0 < |x| < 1 \text{ and } \frac{1}{x} \notin \mathbb{Z}_+ \}$

4. Let $X = \{a, b, c, d, e\}$. Give three separate topologies $\mathcal{T}_1$, $\mathcal{T}_2$, and $\mathcal{T}_3$ on X so that none of them is comparable with the other two.

5. Let $\mathbb{R}_\ell$ denote the real numbers with the half-open (lower limit) topology (see problem 1), and let $\mathbb{R}$ denote the real numbers with the standard topology (again, see problem 1).

   (a) If L is a straight line in the plane, describe the topology that L inherits as a subspace of $\mathbb{R}_\ell \times \mathbb{R}$. Is it a topology we've already seen?

   (b) If L is a straight line in the plane, describe the topology that L inherits as a subspace of $\mathbb{R}_\ell \times \mathbb{R}_\ell$. Is it a topology we've already seen?

   [Hint: For this problem, you might want to consider the following cases separately: 1) L is a vertical line, 2) L is a horizontal line, and 3) L is neither vertical nor horizontal.]

6. Consider the set of real numbers $\mathbb{R}$, and let K denote the set of all numbers of the form $1/n$ for $n \in \mathbb{Z}_+$. Define $\mathcal{B}''$ to be the collection of all open intervals (a, b) in $\mathbb{R}$ along with all sets of the form $(a, b) - K$. $\mathcal{B}''$ is a basis for a topology, and we will let $\mathcal{T}_k$ denote the topology on $\mathbb{R}$ generated by $\mathcal{B}''$. Let $\mathcal{T}_\ell$ be the half-open topology on $\mathbb{R}$ as in problem $1(b)$. Show that $\mathcal{T}_K$ and $\mathcal{T}_\ell$ are not comparable.