Homework 6, Topology

Due Wednesday December 5, 2007

Justify your answers.

1. Let $f_1, f_2 : X \to Y$ be homotopic, continuous maps between topological spaces X and Y, and let $g_1, g_2 : Y \to Z$ be homotopic, continuous maps between the topological spaces Y and Z. Show that $g_1 \circ f_1$ and $g_2 \circ f_2$ are homotopic.

2. Given two topological spaces X and Y, let $[X, Y]$ denote the set of homotopy classes of maps of X into Y.

   (a) Let $I = [0, 1]$ the unit interval. Show that for any topological space X, $[X, I]$ has a single element.

   (b) Show that if Y is path connected, the set $[I, Y]$ has a single element.

3. A topological space X is said to be **contractible** if the identity map $i_x : X \to X$ is nullhomotopic, i.e. the identity map is homotopic to a constant map.

   (a) Show that $\mathbb{R}^n$ is contractible. [Hint: Write down a formula for a homotopy from the identity map to the constant map $g : \mathbb{R}^n \to \mathbb{R}^n$, $g(x) = 0$ where 0 is the origin.]

   (b) Show that a contractible space is path connected.

   (c) Show that if Y is contractible, then for any topological space X, the set $[X, Y]$ has a single element.

   (d) Show that if X is contractible, and Y is path connected, then $[X, Y]$ has a single element.

4. A subset A of $\mathbb{R}^n$ is said to be **star convex** if for some point a_0 of A, all line segments joining a_0 to other points of A lie in A.

   (a) Find a star convex set that is not convex. You can draw a picture if you want to.

   (b) Show that if A is star convex, then A is simply connected.