Exam 2 **Calc III, Section 01**

March 24, 2009

Rules for the Exam:

- No calculators are permitted for the exam.
- No books or notes can be used during the exam.
- Clearly mark your final answer for every problem.
- Show all of your work.

1. (a) (5 points) Evaluate the following integral

$$\int\int_R e^{x+y}\,dA$$

where $R = \{\,(x, y)\,|\,0 \le x \le 3, 2 \le y \le 5\,\}$.

Solution: This is an integral of a continuous function over a rectangular region, so we can use Fubini's Theorem. This gives

$$\int\int_R e^{x+y}\,dA = \int_0^3\int_2^5 e^{x+y}\,dy\,dx = \int_2^5\int_0^3 e^{x+y}\,dx\,dy.$$

Either way you want to do it, we have an iterated integral and we get

$$\begin{aligned}\int_0^3\int_2^5 e^{x+y}\,dy\,dx &= \int_0^3 \left[e^{x+y}|_{y=2}^{y=5}\right]dx \\ &= \int_0^3 \left(e^{x+5} - e^{x+2}\right)dx \\ &= \left(e^{x+5} - e^{x+2}\right)|_{x=0}^{x=3} \\ &= (e^8 - e^5) - (e^5 - e^2) \\ &= e^8 - 2e^5 + e^2 = (e^4 - e)^2.\end{aligned}$$

(b) (15 points) Evaluate the following integral

$$\int\int_D (2x + 1)\sin(x + y)\,dA$$

where D is the region of the xy-plane bounded by $y = x^2$, the line $x = \pi$, and the x-axis.

Solution: The first thing we need to do is to graph the region of integration so that we know how to set up the integral. After doing this, we can set up the integral in one of two ways.

$$\int\int_D (2x + 1)\sin(x + y)\,dA = \int_0^{\pi^2}\int_{\sqrt{y}}^{\pi} (2x + 1)\sin(x + y)\,dx\,dy,$$

or

$$\int\int_D (2x + 1)\sin(x + y)\,dA = \int_0^{\pi}\int_0^{x^2} (2x + 1)\sin(x + y)\,dy\,dx.$$

The second integral is the easier to evaluate, so we will solve that one.

$$\begin{aligned}\int\int_D (2x + 1)\sin(x + y)\,dA &= \int_0^{\pi}\int_0^{x^2} (2x + 1)\sin(x + y)\,dy\,dx \\ &= \int_0^{\pi}\left(-(2x + 1)\cos(x + y)|_{y=0}^{y=x^2}\right)dx \\ &= \int_0^{\pi}\left(-(2x + 1)\cos(x + x^2) + (2x + 1)\cos(x + 0)\right)dx \\ &= -\int_0^{\pi}(2x + 1)\cos(x + x^2)\,dx + \int_0^{\pi}(2x + 1)\cos(x)\,dx.\end{aligned}$$

The first integral can be solved with a simple u substitution, $u = x^2 + x$ and $du = (2x+1)\,dx$. This gives

$$\begin{aligned} -\int_0^{\pi} (2x+1)\cos(x+x^2)\,dx &= -\int_0^{\pi^2+\pi} \cos(u)\,du \\ &= -\sin(u)|_{u=0}^{u=\pi^2+\pi} \\ &= -\sin(\pi^2+\pi)+\sin(0) = -\sin(\pi^2+\pi). \end{aligned}$$

To solve the second integral we will need to use integration by parts with $u = (2x+1)$ and $dv = \cos(x)\,dx$. So $du = -2\,dx$ and $v = \sin(x)$. This gives

$$\begin{aligned} \int_0^{\pi} (2x+1)\cos(x)\,dx &= ((2x+1)\sin(x))\,|_{x=0}^{x=\pi} - \int_0^{\pi} 2\sin(x)\,dx \\ &= ((2\pi+1)\sin(\pi) - (2(0)+1)\sin(0)) - (-2\cos(x)|_{x=0}^{x=\pi}) \\ &= 0 - (-2\cos(\pi) + 2\cos(0)) = -4. \end{aligned}$$

Putting all of this together, we get

$$\int\int_D (2x+1)\sin(x+y)\,dA = \int_0^{\pi}\int_0^{x^2} (2x+1)\sin(x+y)\,dy\,dx = -\sin(\pi^2+\pi) - 4.$$

2. (10 points) Sketch the domain of integration and change the order of integration in the following integral

$$\int_1^2 \int_0^{\ln x} f(x,y)\,dy\,dx.$$

Solution: The first step is to sketch the region of integration so that we can change the order of integration. Changing the order of integration gives

$$\int_1^2 \int_0^{\ln x} f(x,y)\,dy\,dx = \int_0^{\ln 2} \int_{e^y}^{2} f(x,y)\,dx\,dy.$$

3. Let $f(x,y) = e^{y^2}\cos(\pi x) - \sin(xy)$ where

$$x(s,t) = \ln(st), \quad \text{and} \quad y(s,t) = \frac{s-st}{s+t}$$

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(a) (10 points) Find $\frac{\partial f}{\partial t}|_{s=1,t=1}$.

Solution: To do this we need to find the partial derivatives of f which are

$$\frac{\partial f}{\partial x} = e^{y^2}(-\pi\sin(\pi x)) - y\cos(xy) = -\pi e^{y^2}\sin(\pi x) - y\cos(xy),$$

and

$$\frac{\partial f}{\partial y} = 2ye^{y^2}\cos(\pi x) - x\cos(xy).$$

Next, we need to find the derivatives with respect to t of x and y. These are

$$\frac{\partial x}{\partial t} = \frac{1}{t} \quad \text{and} \quad \frac{\partial y}{\partial t} = \frac{-s}{s+t} - \frac{s-st}{(s+t)^2}.$$

Using the formula

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

we have

$$\frac{\partial f}{\partial t} = \left(-\pi e^{y^2}\sin(\pi x) - y\cos(xy)\right)\left(\frac{1}{t}\right) + \left(2ye^{y^2}\cos(\pi x) - x\cos(xy)\right)\left(\frac{-s}{s+t} - \frac{s-st}{(s+t)^2}\right).$$

Now we need to evaluate this at $s = 1, t = 1$. Notice that if $s = 1$ and $t = 1$, then $x = 0$ and $y = 0$. Therefore, we get

$$\frac{\partial f}{\partial t}|_{s=1,t=1} = (0-0)\left(\frac{1}{1}\right) + (0-0)\left(\frac{-1}{2} - \frac{0}{(2)^2}\right) = 0.$$

(b) (10 points) Find $\frac{\partial f}{\partial s}$. Your answer can be in terms of x, y, s and t.

Solution: For this, we use the formula

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}.$$

With what we have already done, we only need to find the derivatives of x and y with respect to s. These are

$$\frac{\partial x}{\partial s} = \frac{1}{s} \quad \text{and} \quad \frac{\partial y}{\partial s} = \frac{1-t}{s+t} - \frac{s-st}{(s+t)^2}.$$

So altogether, we have

$$\frac{\partial f}{\partial s} = \left(-\pi e^{y^2}\sin(\pi x) - y\cos(xy)\right)\left(\frac{1}{s}\right) + \left(2ye^{y^2}\cos(\pi x) - x\cos(xy)\right)\left(\frac{1-t}{s+t} - \frac{s-st}{(s+t)^2}\right).$$

4. (10 points) Let $F(x,y,z) = 3xy - z^2 + 4xy^2 - y^2$ and $\vec{v} =< 1, 1, -4 >$. Find the directional derivative of F in the direction of $\vec{v}$ at the point$(2,1,3)$.

Solution: To find the directional derivative, we can use the gradient

$$\vec{\nabla} F =< \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} >=< 3y + 4y^2, 3x + 8xy - 2y, -2z > .$$

To use the formula, we need to be taking the directional derivative in the direction of a unit vector. So we need to find the unit vector in the direction of $\vec{v}$. Calculating the length of $\vec{v}$ we get $\sqrt{1^2 + 1^2 + (-4)^2} = \sqrt{18}$. So the unit vector in the direction of $\vec{v}$ is

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} =< \frac{1}{\sqrt{18}}, \frac{1}{\sqrt{18}}, \frac{-4}{\sqrt{18}} > .$$

Applying the formula for the directional derivative, we get

$$\begin{aligned} D_{\vec{u}}F(2,1,3) &= \vec{\nabla} F(2,1,3) \cdot \vec{u} \\ &= < 7, 20, -6 > \cdot < \frac{1}{\sqrt{18}}, \frac{1}{\sqrt{18}}, \frac{-4}{\sqrt{18}} > \\ &= \frac{7}{\sqrt{18}} + \frac{20}{\sqrt{18}} + \frac{24}{\sqrt{18}} = \frac{51}{\sqrt{18}} = \frac{17}{\sqrt{2}}. \end{aligned}$$

5. (10 points) Find the equation for the tangent plane to the surface

$$3xy^2 - 8yz + 4xz^2 + y^2 = 4$$

at the point $P(1,1,2)$.

Solution: To find the tangent plane to a level surface given by the equations $F(x,y,z) = k$ where k is a constant, we can use the gradient

$$\vec{\nabla} F =< \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} > .$$

In this case, we have

$$\vec{\nabla} F =< 3y^2 + 4z^2, 6xy - 8z + 2y, -8y + 8xz > .$$

The gradient at the point $(1,1,2)$ is perpendicular to the tangent plane to the surface at the point $(1,1,2)$. So we have that the tangent plane goes through the point $(1,1,2)$ and is perpendicular to the vector $\vec{\nabla} F(1,1,2) =< 19, -8, 8 >$. Therefore, using our formula for the equation of a plane, we get

$$19(x-1) - 8(y-1) + 8(z-2) = 0$$

or

$$19x - 8y + 8z = 27.$$

6. (15 points) Find all the relative maxima, minima and saddle points of the function

$$f(x, y) = x^2y - 6y^2 - 3x^2.$$

Solution: For this problem, we need to find the critical points of the function. So we need the partial derivatives

$$\frac{\partial f}{\partial x} = 2xy - 6x \quad \text{and} \quad \frac{\partial f}{\partial y} = x^2 - 12y.$$

So the critical points are going to come at the solutions to the system of equations

$$\begin{aligned} 2xy - 6x &= 0 \\ x^2 - 12y &= 0 \end{aligned}$$

Consider the first equation and we get $2x(y-3) = 0$. This gives us either $x = 0$ or $y = 3$. If $x = 0$, then the second equation gives $y = 0$. If $y = 3$, then the second equation gives $x^2 = 36$ or $x = \pm 6$. Therefore, we have three critical points $(0, 0)$, $(6, 3)$ and $(-6, 3)$.

To figure out if each point is a maximum, minimum or saddle point, we need to use the second derivative test. To do this, we need to calculate

$$\begin{aligned} D &= \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 \\ &= (2y-6)(-12) - (2x)^2 = -24y + 72 - 4x^2. \end{aligned}$$

Now we need to check the value od D at each of the critical points:

- For the point $(0, 0)$, we have $D(0, 0) = 72 > 0$. So next we need to check the value of $f_{xx}(0, 0) = -6 < 0$. This means that the point $(0, 0)$ is a local maximum for the function.
- For the point $(6, 3)$, we have $D(6, 3) = -72 + 72 - 144 = -144 < 0$. Therefore, the point $(6, 3)$ is a saddle point for the funciton.
- For the point $(-6, 3)$, we have $D(-6, 3) = -72 + 72 - 144 = -144 < 0$. Therefore, the point $(-6, 3)$ is a saddle point for the funciton.

Since there aren't any other critical points, we have found all of the relative maxima, minima and saddle points of the function.

7. (15 points) Let $f(x, y) = x^2 + y^2$. Find the absolute maximum and absolute minimum values of $f(x, y)$ on the domain

$$D = \{\, (x, y) \,|\, x^2 + xy + y^2 \leq 3 \,\}.$$

Solution: There are two steps for this problem. First we need to find the critical points of the function and the value of the function at all of the critical points. Then we need to find the extreme values on the boundary of the domain. All of these will give us possibilities for the absolute maximum and minimum values of the function on the domain.

First, we find the critical points of the function, which means that we have to find the first partial derivatives

$$\frac{\partial f}{\partial x} = 2x \quad \text{and} \quad \frac{\partial f}{\partial y} = 2y.$$

So we need to solve the following system of equations

$$\begin{aligned} 2x &= 0 \\ 2y &= 0. \end{aligned}$$

The only solution to the system is the point $(0, 0)$. So there is only one critical point, and $f(0, 0) = 0$.

Next, we need to find the extreme values of the function on the boundary of the domain. In this case, the boundary of the domain is given by the curve $x^2 + xy + y^2 = 3$. So we need to find the extreme values of the function $f(x, y) = x^2 + y^2$ given the condition $x^2 + xy + y^2 = 3$. This is a Lagrange multiplier problem., adn so we need to solve the following system of three equations

$$\begin{aligned} 2x &= \lambda(2x + y) && (1) \\ 2y &= \lambda(2y + x) && (2) \\ x^2 + xy + y^2 &= 3 && (3) \end{aligned}$$

To solve this system, we can multiply both sides of equation (1) by y and multiply both sides of equation (2) by x. This gives

$$\begin{aligned} 2xy &= \lambda(2xy + y^2) \\ 2xy &= \lambda(2xy + x^2) \end{aligned}$$

Setting the right sides equal to each other, we end up with $\lambda x^2 = \lambda y^2$. This gives us two possibilities: either $\lambda = 0$ or $x^2 = y^2$.

If $\lambda = 0$, then equation (1) tells us that $x = 0$, and equation (2) tells us that $y = 0$. But $x = 0$ and $y = 0$ violates equation (3). Therefore, we can't have $\lambda = 0$. This only leaves the possibility $x^2 = y^2$. Solving this gives us two cases: $x = y$ or $x = -y$.

- If $x = y$, then equation (3) becomes $y^2 + y^2 + y^2 = 3$ or $3y^2 = 3$. This gives $y^2 = 1$ or $y = \pm 1$. Since we know $x = y$, this gives us two points $(1, 1)$ and $(-1, -1)$.
- If $x = -y$, then equation (3) becomes $(-y)^2 + (-y)y + y^2 = 3$ which simplifies to $y^2 = 3$. This gives $y = \pm\sqrt{3}$. Since $x = -y$, we again get two points $(-\sqrt{3}, \sqrt{3})$ and $(\sqrt{3}. -\sqrt{3})$.

Getting everything together, we have the following: $f(0, 0) = 0$, $f(1, 1) = 2$, $f(-1, -1) = 2$, $f(-\sqrt{3}, \sqrt{3}) = 6$ and $f(\sqrt{3}, -\sqrt{3}) = 6$. Therefore, the absolute minimum value of the function on the domain is 0, and the absolute maximum value of the function on the domain is 6.