Solutions for Exam 3 Calc III, Section 01

1. (10 points) Evaluate the line integral

$$\int_C x^{3/2} e^y \, ds$$

where C is the piece of the curve $x = e^{2y}$ going from $y = 3$ to $y = 5$.

Solution: To evaluate the line integral, we need to find a parametrization for the curve. A simple way to paramatrize the curve is to parametrize it in terms of y, which gives

$$\vec{\mathbf{r}}(y) = < e^{2y}, y > \quad \text{for } 3 \leq y \leq 5.$$

Using this parametrization, we get

$$\begin{aligned} \int_C x^{3/2} e^y \, ds &= \int_3^5 (e^{2y})^{3/2} e^y \sqrt{(2e^{2y})^2 + 1} \, dy \\ &= \int_3^5 e^{4y} \sqrt{4e^{4y} + 1} \, dy. \\ &= \frac{1}{16} \int_{4e^{12}+1}^{4e^{10}+1} \sqrt{u} \, du \\ &= \frac{1}{16} \frac{2}{3} u^{3/2} |_{u=4e^{12}+1}^{u=4e^{10}+1} \\ &= \frac{1}{24}(4e^{20}+1)^{3/2} - \frac{1}{24}(4e^{12}+1)^{3/2}. \end{aligned}$$

2. (10 points) Let R be the region bounded by $xy = 1$, $xy = 2$, $xy^2 = 2$, and $xy^2 = 4$. Use the substitution $x = \frac{u^2}{v}$, $y = \frac{v}{u}$. **Set up but do not solve** an integral to compute

$$\int \int_R (xy^2 - xy) \, dA$$

using this transformation.

Solution: To use this substitution, we will need to change the region R to be in terms of u and v. Also, we will need to calculate the Jacobian for this transformation. First, let's consider the transformation. Since $x = \frac{u^2}{v}$ and $y = \frac{v}{u}$ we get $xy = u$ and $xy^2 = v$. Therefore, the limits of the region turn into $u = 1$, $u = 2$, $v = 2$, and $v = 4$. So after the transformation, we have a rectangular region in u and v.

Next, we need to determine the Jacobian for this transformation. The Jacobian will be given as the determinant of a matrix

$$\begin{aligned} \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} &= \begin{vmatrix} \frac{2u}{v} & \frac{-u^2}{v^2} \\ \frac{-v}{u^2} & \frac{1}{u} \end{vmatrix} \\ &= \left(\frac{2u}{v}\right)\left(\frac{1}{u}\right) - \left(\frac{-v}{u^2}\right)\left(\frac{-u^2}{v^2}\right) \\ &= \frac{2}{v} - \frac{1}{v} = \frac{1}{v}. \end{aligned}$$

Putting all of this together, we get that the integral we stared with turns into

$$\int_1^2 \int_2^4 (v - u) \frac{1}{v} \, dv \, du.$$

3. Determine if the following vector fields are conservative. If they are, find their potential functions.

(a) (10 points) $\vec{\mathbf{F}}(x, y) =< e^x \cos(y), e^x \sin(y) >$.
Solution: To determine if the vector field is conservative, we need to check if $\partial P/\partial y$ is equal to $\partial Q/\partial x$. In this case, we have $P(x, y) = e^x \cos(y)$ and $Q(x, y) = e^x \sin(y)$. therefore

$$\frac{\partial P}{\partial y} = -e^x \sin(y) \quad \text{and} \quad \frac{\partial Q}{\partial x} = e^x \sin(y).$$

Since they are not equal, the vector field is not conservative.

(b) (10 points) $\vec{\mathbf{F}}(x, y) =< 12xy + y\sin(x), 6x^2 + 30y - \cos(x) >$.
Solution: Just as above, we need to check if $\partial P/\partial y$ is equal to $\partial Q/\partial x$. Here $P(x, y) = 12xy + y\sin(x)$ and $Q(x, y) = 6x^2 + 10y - \cos(x)$, and so

$$\frac{\partial P}{\partial y} = 12x + \sin(x) \quad \text{and} \quad \frac{\partial Q}{\partial x} = 12x + \sin(x).$$

Since they are equal, the vector field is conservative.

Since the vector field is conservative, we now need to find the potential functon which satisfies $\vec{\mathbf{F}} = \vec{\nabla} f$. To do this, we notice that

$$\frac{\partial f}{\partial x} = 12xy + y\sin(x) \quad \text{and} \quad \frac{\partial f}{\partial y} = 6x^2 + 30y - \cos(x).$$

Integrating $\partial f/\partial x$ with respect to x we get

$$f(x, y) = 6x^2 y - y\cos(x) + g(y)$$

. Using this, we get $\partial f/\partial y = 6x^2 - \cos(x) + g'(y)$. Comparing this to what we already know, we get $g'(y) = 30y$, and therefore $g(y) = 15y^2 + K$ where K is a constant.
Combining all of this together, we get

$$f(x, y) = 6x^2 y - y\cos(x) + 15y^2 + K.$$

4. (15 points) Evaluate the integral

$$\int\int\int_E (xyz)\, dV$$

where E is the solid region bounded by the cylinder $x^2 + z^2 = 4$, the plane $y = 2$, the plane $y = z$, and the plane $z = 0$.

Solution: The first thing to do is to determine what the region E looks like. After you do that, you will see that there are a few ways to set up the integral, either

$$\int_{-2}^{2} \int_{0}^{\sqrt{4-x^2}} \int_{z}^{2} xyz\, dy\, dz\, dx$$

or

$$\int_{0}^{2} \int_{z}^{2} \int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}} xyz\, dx\, dy\, dz$$

. These are the two main ways to do it. The important thing is to see that you can't integrate dz first over this domain. Pick one of the integrals and evaluate and you get

$$\begin{aligned}
\int_{-2}^{2} \int_{0}^{\sqrt{4-x^2}} \int_{z}^{2} xyz\, dy\, dz\, dx &= \int_{-2}^{2} \int_{0}^{\sqrt{4-x^2}} \frac{1}{2}(xz)(2^2 - z^2)\, dz\, dx \\
&= \int_{-2}^{2} \int_{0}^{\sqrt{4-x^2}} 2xz - \frac{1}{2}xz^3\, dz\, dx \\
&= \int_{-2}^{2} xz^2 - \frac{1}{8}xz^4|_{z=0}^{z=\sqrt{4-x^2}}\, dx \\
&= \int_{-2}^{2} x(4 - x^2) - \frac{1}{8}x(4 - x^2)^2\, dx \\
&= \frac{-1}{4}(4 - x^2)^2 + \frac{1}{48}(4 - x^2)^3|_{x=-2}^{x=2} = 0.
\end{aligned}$$

The last line was gotten by using the substitution $u = 4 - x^2$ and $du = -2x\, dx$.

5. (10 points) Find the volume under the plane $z = y$ and above the region of the xy-plane in the first quadrant bounded by the curve $x^2 + y^2 = 6$, the curve $y = x^2$, and the y-axis.

Solution: To do this we just need to set up a double integral. The first step is to figure out what the domain on the xy-plane will look like. To do that, we need to know where the curves $x^2 + y^2 = 6$ and $y = x^2$ intersect. So we need to solve the simultanious system of equations, which gives $y + y^2 = 6$. This turns into $y^2 + y - 6 = 0$ or $(y-2)(y+3) = 0$. Therefore, $y = 2$ (since y is positive from $y = x^2$ we can ignore the solution $y = -3$). Since $y = 2$, we get $x = \sqrt{2}$ (again, since x is positive, we can ignore $x = -\sqrt{2}$). Using this we can see that the domain of integration can be described by $0 \le x \le \sqrt{2}$ and $x^2 \le y \le \sqrt{6 - x^2}$. So we have

$$\begin{aligned}
\int_0^{\sqrt{2}} \int_{x^2}^{\sqrt{6-x^2}} y\,dy\,dx &= \int_0^{\sqrt{2}} \frac{1}{2} y^2 |_{y=x^2}^{y=\sqrt{6-x^2}}\,dx \\
&= \int_0^{\sqrt{2}} \frac{1}{2}(6 - x^2) - \frac{1}{2} x^4\,dx \\
&= \left(3x - \frac{1}{6}x^3 - \frac{1}{10}x^5\right) |_{x=0}^{x=\sqrt{2}} \\
&= 3\sqrt{2} - \frac{2\sqrt{2}}{6} - \frac{4\sqrt{2}}{10}.
\end{aligned}$$

You can also solve this by setting up a triple integral, but you will get the same answer.

6. (15 points) Let $\vec{\mathbf{F}}(x, y) =< \frac{2x}{x^2 + e^y}, \frac{e^y}{x^2 + e^y} >$. Compute the integral

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$$

where C is the curve given by

$$\vec{\mathbf{r}}(t) =< \ln(3t^2 + 1)\sin(\frac{t\pi}{2}),\ e^{t^2 - 6t + 8}\cos(\frac{t\pi}{2}) >$$

for $2 \le t \le 4$.

Solution: Whenever you see the line integral of a vector field over a curve, the first thing you should do is check if the vector field is conservative. In this case, we see that

$$\frac{\partial P}{\partial y} = \frac{-2xe^y}{(x^2 + e^y)^2} \quad \text{and} \quad \frac{\partial Q}{\partial x} = \frac{-2xe^y}{(x^2 + e^y)^2}.$$

This shows that the vector field is conservative. Therefore, we need to find the potential $f(x, y)$ with $\vec{\mathbf{F}} = \vec{\nabla} f$. By calculation, we can get

$$f(x, y) = \ln(x^2 + e^y) + K.$$

Using this, we see that the integral we want to solve turn into (using the fundamental theorem of calculus for line integrals)

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_C \vec{\nabla} f \cdot d\vec{\mathbf{r}} = f(\vec{\mathbf{r}}(b)) - f(\vec{\mathbf{r}}(a)),$$

where the curve C is given by $\vec{\mathbf{r}}(t)$ for $a \le t \le b$. So in this case, we have

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = f(\vec{\mathbf{r}}(4)) - f(\vec{\mathbf{r}}(2)) = f(0, 1) - f(0, -1) = \ln(0 + e^1) - \ln(0 + e^{-1}) = 1 - (-1) = 2.$$

7. (10 points) Use spherical coordinates to evaluate the integral

$$\int\int\int_E z^2\,dV$$

where E is the solid hemisphere above the plane $z = 0$ bounded by $x^2 + y^2 + z^2 = 1$.

Solution: Spherical coordinates are given by $x = \rho\cos\theta\sin\phi$, $y = \rho\sin\theta\sin\phi$, and $z = \rho\cos\phi$. The solid hemispherical region above the plane $z = 0$ is given by $0 \le \rho \le 1$, $0 \le \theta \le 2\pi$, and $0 \le \phi \le \pi/2$. Remembering the correction term for integrating in spherical coordiantes, we have

$$\begin{aligned}
\int\int\int_E z^2\,dV &= \int_0^{\pi/2}\int_0^{2\pi}\int_0^1 (\rho\cos\phi)^2\rho^2\sin\phi\,d\rho\,d\theta\,d\rho \\
&= \int_0^{\pi/2}\int_0^{2\pi}\int_0^1 rho^4\cos^2\phi\sin\phi\,d\rho\,d\theta\,d\rho \\
&= \int_0^{\pi/2}\int_0^{2\pi}\frac{1}{5}\cos^2\phi\sin\phi\,d\theta\,d\phi \\
&= \int_0^{\pi/2}\frac{2\pi}{5}\cos^2\phi\sin\phi\,d\phi \\
&= \frac{2\pi}{15}(-\cos^3(\pi/2)+\cos^3(0)) = \frac{2\pi}{15}.
\end{aligned}$$

8. (10 points) Use Green's Theorem to evaluate the line integral

$$\int_C \left(3y - e^{\sin(x)}\right)\,dx + \left(7x + \sqrt{y^4+1}\right)\,dy$$

where C is the curve given by the piece of $y = x^3$ from $x = 0$ to $x = 1$ followed by the line segment from $(1,1)$ to $(0,0)$.

Solution: This is a simple application of Green's Theorem. We do need to get a good idea of the domain bounded by the curve C to apply the theorem. If you dra a picture of the domain, you will see that the domain is bounded by $0 \le x \le 1$ and $x^3 \le y \le x$. Therefore, Green's Theorem gives us

$$\begin{aligned}
\int_C \left(3y - e^{\sin(x)}\right)\,dx + \left(7x + \sqrt{y^4+1}\right)\,dy &= \int\int_D 7 - 3\,dA \\
&= \int_0^1\int_{x^3}^{x} 4\,dy\,dx \\
&= \int_0^1 4x - x^3\,dx \\
&= 2x^2 - x^4|_{x=0}^{x=1} = 2 - 1 = 1.
\end{aligned}$$