Homework 4, Calc III
Solutions

1. Find the domain of the function and sketch it on the xy-plane.

   (a) $f(x, y) = \sqrt{xy}$.
   Solution: For this function, the domain is going to consist of all points (x, y) satisfying $xy \geq 0$ because we need anything in a square root to be greater than or equal to 0. This gives us that the domain will be the first and third quadrants of the xy-plane including both the x-axis and the y-axis. There is a picture on a seperate sheet.

   (b) $f(x, y) = \sqrt{y^2 - x^2}\ln(y + x)$.
   Solution: For this function, there are two conditions that have to be satisfied. Because of the square root, we need $y^2 - x^2 \geq 0$ or $(y - x)(y + x) \geq 0$. Because of the logarithm, we need $y + x > 0$. Combining these two considerations, we see that since $y + x > 0$ we must have $y - x \geq 0$. Therefore we have that both $y > -x$ and $y \geq x$. Together these give a triangle of area above the x-axis on the xy-plane which is between the line $y = x$ and the line $y = -x$. The domain includes the piece of the line $y = x$ but doesn't include the piece of the line $y = -x$. There is a picture on a seperate sheet.

2. Find the limit or explain why it doesn't exist

   (a)
$$\lim_{(x,y)\to(0,0)} \frac{x^2\sin^2(y)}{x^2 + y^2}.$$

   Solution: The first thing to do with a limit like this is to try to input the values into the function. In this case when we plug $(0, 0)$ into the expression, we get

$$\frac{0^2\sin^2(0)}{0^2 + 0^2} = \frac{0}{0}$$

   which is not an allowed answer since we can't divide by 0. Therefore, this function is not continuous at this point, so we need to try other methods.
   Since we have (x, y) approaching the point $(0, 0)$, we can set $x = 0$ and see what happens as y approaches 0. This means considering the limit along the y-axis. This gives

$$\lim_{(0,y)\to(0,0)} \frac{0\sin^2(y)}{0^2 + y^2} = \lim_{(0,y)\to(0,0)} \frac{0}{y^2} = 0$$

   Similarly, if we approach along the x-axis by setting $y = 0$, we will get

$$\lim_{(x,0)\to(0,0)} \frac{x^2\sin^2(0)}{x^2 + 0^2} = \lim_{(x,0)\to(0,0)} \frac{0}{x^2} = 0.$$

   So far they have both given us the same answer. In fact, if we consider $y = mx$ for m any constant (that is, approaching $(0, 0)$ on a straight line), we will get the same answer. The same answer comes up for $y = x^2$, $x = y^2$ and many other possibilities. However, this is not proof that the limit exists. To prove the limit exists and is indeed 0, we need to use another tool.
   Consider that $\frac{x^2}{x^2 + y^2} \leq 1$ for all values of (x, y) near $(0, 0)$. Combine this with the fact

$$0 \leq \sin^2(y)$$

   to get

$$0 \leq \frac{x^2\sin^2(y)}{x^2 + y^2} \leq \sin^2(y).$$

   Using the squeeze theorem, we can see that

$$0 = \lim_{(x,y)\to(0,0)} 0 \leq \lim_{(x,y)\to(0,0)} \frac{x^2\sin^2(y)}{x^2 + y^2} \leq \lim_{(x,y)\to(0,0)} \sin^2(y) = 0.$$

   So therefore, the limit exists and is 0.

(b)

$$\lim_{(x,y)\to(\pi,0)} \frac{xy}{y-\cos(x)}.$$

Solution: First we check to see if we can plug in the values. In this case, if we plug in the values we get

$$\frac{\pi 0}{0-\cos(\pi)} = \frac{0}{1} = 0.$$

Therefore, this function is continuous at the point, and the limit is 0.

3. Find all the first partial derivatives of the functions

(a)

$$f(x,y) = e^{2xy} - \sin(x)\cos(x+y).$$

Solution: To take the partial derivative with respect to x we treat y as a constant. Therefore, we get

$$\frac{\partial f}{\partial x} = 2ye^{2xy} - \cos(x)\cos(x+y) + \sin(x)\sin(x+y).$$

Similarly to get the partial derivative with respect to y we treat x as a constant. This gives

$$\frac{\partial f}{\partial y} = 2xe^{2xy} + \sin(x)\sin(x+y).$$

(b)

$$f(x,y) = \arcsin(x\sqrt{y}).$$

Solution: Just as in the previous case, we need to take the first partial derivatives by treating the other variable as a constant. We get

$$\frac{\partial f}{\partial x} = \frac{\sqrt{y}}{\sqrt{1-x^2y}},$$

and

$$\frac{\partial f}{\partial y} = \frac{x}{2\sqrt{y(1-x^2y)}}.$$

4. Find all the second partial derivatives of

$$f(x,y) = \frac{xy}{x^2-y}.$$

Solution: To find the second partial derivatives, we need to first find the first partial derivatives. We have

$$\frac{\partial f}{\partial x} = \frac{y}{x^2-y} - \frac{xy}{(x^2-y)^2}(2x) = \frac{(x^2-y)y - 2x^2y}{(x^2-y)^2} = \frac{-y(x^2+y)}{(x^2-y)^2}.$$

and

$$\frac{\partial f}{\partial y} = \frac{x}{x^2-y} - \frac{xy}{(x^2-y)^2}(-1) = \frac{(x^2-y)x + xy}{(x^2-y)^2} = \frac{x^3}{(x^2-y)^2}.$$

Now we need to find the following four second derivatives

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right), \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right), \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right).$$

We have

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{-y(x^2+y)}{(x^2-y)^2}\right) = \frac{-2xy}{(x^2-y)^2} - 2\frac{-y(x^2+y)}{(x^2-y)^3}(2x) = \frac{(x^2-y)(-2xy) + 4xy(x^2+y)}{(x^2-y)^3} = \frac{2x^3y + 6xy^2}{(x^2-y)^3},$$

and

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{x^3}{(x^2-y)^2}\right) = \frac{3x^2}{(x^2-y)^2} - 2\frac{x^3}{(x^2-y)^3}(2x) = \frac{(x^2-y)(3x^2) - 4x^4}{(x^2-y)^3} = \frac{-x^4 - 3x^2y}{(x^2-y)^3},$$

and

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{-y(x^2+y)}{(x^2-y)^2}\right) = \frac{-x^2-2y}{(x^2-y)^2} - 2\frac{-y(x^2+y)}{(x^2-y)^3}(-1) = \frac{(x^2-y)(-x^2-2y)-2y(x^2+y)}{(x^2-y)^3} = \frac{-x^4-3x^2y}{(x^2-y)^3},$$

and

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{x^3}{(x^2-y)^2}\right) = -2\frac{x^3}{(x^2-y)^3}(-1) = \frac{2x^3}{(x^2-y)^3}.$$

Notice that we have

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right).$$

Why isn't this a surprise?

5. Find the linearization $L(x, y)$ of the function at the given point

(a)

$$f(x, y) = \frac{x^2-4y}{x+3y} \quad P(2, 2).$$

Solution: In order to find the linearization, we need to find the first partial derivatives of the function.

$$\frac{\partial f}{\partial x} = \frac{2x}{x+3y} - \frac{(x^2-4y)}{(x+3y)^2}(1) = \frac{(x+3y)(2x)-(x^2-4y)}{(x+3y)^2} = \frac{x^2+6xy+4y}{(x+3y)^2},$$

and

$$\frac{\partial f}{\partial y} = \frac{-4}{x+3y} - \frac{x^2-4y}{(x+3y)^2}(3) = \frac{(x+3y)(-4)-3(x^2-4y)}{(x+3y)^2} = \frac{-4x-3x^2}{(x+3y)^2}.$$

Notice that both of the first partial derivatives are defined at and contiuous near the point $P(2, 2)$. Therefore, we can apply the formula for the linearization, which is,

$$L(x, y) = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0),$$

to get

$$L(x, y) = -\frac{1}{2} + \frac{9}{16}(x-2) - \frac{5}{16}(y-2),$$

using that $f(2, 2) = -1/2$, $f_x(2, 2) = 9/16$, and $f_y(2, 2) = -5/16$.

(b)

$$f(x, y) = e^{xy}\sin xy \quad P(0, \pi).$$

Solution: Again, we need to find the first partial derivatives.

$$\frac{\partial f}{\partial x} = ye^{xy}\sin(xy) + ye^{xy}\cos(xy),$$

and

$$\frac{\partial f}{\partial y} = xe^{xy}\sin(xy) + xe^{xy}\cos(xy).$$

Using the same formula from the previous problem, we get

$$L(x, y) = 0 + \pi(x - 0) + 0(y - \pi).$$

or

$$L(x, y) = \pi x.$$