Homework 5 Solutions, Calc III

1. Use the chain rule to find dz/dt.

   (a) $z = \arctan(y/x)$, $x = e^t$, $y = 1 - e^{-t}$.
   Solution: Recall that the chain rule tells us

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

So to use the chain rule, we need to find the partial derivatives of z as well as the derivatives of x and y.

$$\frac{\partial z}{\partial x} = \frac{1}{\left(\frac{y}{x}\right)^2 + 1}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2\left(\frac{y^2}{x^2}+1\right)} = -\frac{y}{y^2+x^2},$$

$$\frac{\partial z}{\partial y} = \frac{1}{\left(\frac{y}{x}\right)^2 + 1}\left(\frac{1}{x}\right) = \frac{1}{x\left(\frac{y^2}{x^2}+1\right)} = \frac{1}{\frac{y^2+x^2}{x}} = \frac{x}{y^2+x^2},$$

$$\frac{dx}{dt} = e^t, \text{ and } \frac{dy}{dt} = e^{-t}.$$

Combining these, we get

$$\frac{dz}{dt} = \left(-\frac{y}{y^2+x^2}\right)e^t + \left(\frac{x}{y^2+x^2}\right)e^{-t},$$

or, if you want to put it only in terms of t,

$$\frac{dz}{dt} = \left(-\frac{1-e^{-t}}{(1-e^{-t})^2+e^{2t}}\right)e^t + \left(\frac{e^t}{(1-e^{-t})^2+e^{2t}}\right)e^{-t} = \frac{2-e^t}{(1-e^{-t})^2+e^{2t}}.$$

   (b) $z = \dfrac{x^2-2xy-y^2}{x^2+y^2}$, $x = \sin(2t^2)$, $y = \cos(2t^2)$.
   Solution: Similarly to the previous problem, we need to find the partials of z and the derivatives of x and y.

$$\frac{\partial z}{\partial x} = \frac{2x-2y}{x^2+y^2} - \frac{x^2-2xy-y^2}{(x^2+y^2)^2}(2x) = \frac{(x^2+y^2)(2x-2y)-2x(x^2-2xy-y^2)}{(x^2+y^2)^2} = \frac{2x^2y+4xy^2-2y^3}{(x^2+y^2)^2}.$$

$$\frac{\partial z}{\partial y} = \frac{-2x-2y}{x^2+y^2} - \frac{(x^2-2xy-y^2}{(x^2+y^2)^2}(2y) = \frac{(x^2+y^2)(-2x-2y)-2y(x^2-2xy-y^2)}{(x^2+y^2)^2} = \frac{2xy^2-2x^3-4x^2y}{(x^2+y^2)^2},$$

$$\frac{dx}{dt} = 4t\cos(2t^2), \text{ and } \frac{dy}{dt} = -4t\sin(2t^2).$$

Altogether, this gives

$$\frac{dz}{dt} = \left(\frac{2x^2y+4xy^2-2y^3}{(x^2+y^2)^2}\right)4t\cos(2t^2) + \left(\frac{2xy^2-2x^3-4x^2y}{(x^2+y^2)^2}\right)(-4t\sin(2t^2)),$$

or if you want to get it only in terms of t,

$$\begin{aligned}
\frac{dz}{dt} &= \frac{2\sin^2(2t^2)\cos(2t^2)+4\sin(2t^2)\cos^2(2t^2)-2\cos^3(2t^2)}{(\sin^2(2t^2)+\cos^2(2t^2))^2}(4t\cos(2t^2)\\
&\quad+\frac{2\sin(2t^2)\cos^2(2t^2)-2\sin^3(2t^2)-4\sin^2(2t^2)\cos(2t^2)}{(\sin^2(2t^2)+\cos^2(2t^2))^2}(-4t\sin(2t^2))\\
&= 8t\sin^2(2t^2)\cos^2(2t^2)+16t\sin(2t^2)\cos^3(2t^2)-8t\cos^4(2t^2)\\
&\quad -8t\sin^2(2t^2)\cos^2(2t^2)+8t\sin^4(2t^2)+16t\sin^3(2t^2)\cos(2t^2),\\
&= 16t\sin(2t^2)\cos^3(2t^2)-8t\cos^4(2t^2)+8t\sin^4(2t^2)+16t\sin^3(2t^2)\cos(2t^2),\\
&= 16t\left[\sin(2t^2)\cos^3(2t^2)+\sin^3(2t^2)\cos(2t^2)\right]+8t\left[\sin^2(2t^2)-\cos^2(2t^2)\right].\\
&= 8t[\sin(4t^2)+\sin^2(2t^2)-\cos^2(2t^2)] = 8t[\sin(4t^2)-\cos(4t^2)].
\end{aligned}$$

The last line is not a typo. Just for fun, figure out why.

2. Use the chain rule to find $\partial z/\partial s$ and $\partial z/\partial t$.

   (a) $z = e^x \sin(y) + \ln(xy)$, $x = \sqrt{s^2+t^2}$, $y = st + s + t$.

   Solution: Here we have a slightly more complicated situation because x and y are both functions of two variables, s and t. That means that for the chain rule here we have the following two formulas.

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s},$$

   and

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}.$$

   So we need to know all the appropriate partial derivatives of z, x and y.

$$\frac{\partial z}{\partial x} = e^x \sin(y) + \frac{y}{xy} = e^x \sin(y) + \frac{1}{x}.$$

$$\frac{\partial z}{\partial y} = e^x \cos(y) + \frac{x}{xy} = e^x \cos(y) + \frac{1}{y}.$$

$$\frac{\partial x}{\partial s} = \frac{s}{\sqrt{s^2+t^2}}, \text{ and } \frac{\partial x}{\partial t} = \frac{t}{\sqrt{s^2+t^2}},$$

$$\frac{\partial y}{\partial s} = t+1, \text{ and } \frac{\partial y}{\partial t} = s+1.$$

   Combining all of this we get,

$$\frac{\partial z}{\partial s} = \left(e^x \sin(y) + \frac{1}{x}\right)\frac{s}{\sqrt{s^2+t^2}} + \left(e^x \cos(y) + \frac{1}{y}\right)(t+1),$$

   and

$$\frac{\partial z}{\partial t} = \left(e^x \sin(y) + \frac{1}{x}\right)\frac{t}{\sqrt{s^2+t^2}} + \left(e^x \cos(y) + \frac{1}{y}\right)(s+1).$$

   (b) $z = \sin(u/v) + \cos(v/u)$, $u = e^{2s} + e^{2t}$, $v = \arctan(s+t)$.

   Solution: Here just like in the previous problem, we need to find all the appropriate partial derivatives of z, u and v.

$$\frac{\partial z}{\partial u} = \cos(u/v)\left(\frac{1}{v}\right) - \sin(v/u)\left(-\frac{v}{u^2}\right) = \frac{\cos(u/v)}{v} + \frac{v\sin(v/u)}{u^2},$$

$$\frac{\partial z}{\partial v} = \cos(u/v)\left(-\frac{u}{v^2}\right) - \sin(v/u)\left(\frac{1}{u}\right) = \frac{-u\cos(u/v)}{v^2} - \frac{\sin(v/u)}{u},$$

$$\frac{\partial u}{\partial s} = 2e^{2s}, \text{ and } \frac{\partial u}{\partial t} = 2e^{2t},$$

$$\frac{\partial v}{\partial s} = \frac{1}{1+(s+t)^2}, \text{ and } \frac{\partial v}{\partial t} = \frac{1}{1+(s+t)^2}.$$

   So putting it all together, we get

$$\frac{\partial z}{\partial s} = \left(\frac{\cos(u/v)}{v} + \frac{v\sin(v/u)}{u^2}\right)(2e^{2s}) + \left(\frac{-u\cos(u/v)}{v^2} - \frac{\sin(v/u)}{u}\right)\left(\frac{1}{1+(s+t)^2}\right),$$

   and

$$\frac{\partial z}{\partial t} = \left(\frac{\cos(u/v)}{v} + \frac{v\sin(v/u)}{u^2}\right)(2e^{2t}) + \left(\frac{-u\cos(u/v)}{v^2} - \frac{\sin(v/u)}{u}\right)\left(\frac{1}{1+(s+t)^2}\right).$$

3. Consider the function

$$f(x,y) = e^{xy}(x+y).$$

(a) Find the gradient $\vec{\nabla} f$ of f.

Solution: To find the gradient, we need to know the first partial derivatives of f.

$$\frac{\partial f}{\partial x} = y(x+y)e^{xy} + e^{xy} = e^{xy}(xy + y^2 + 1),$$

and

$$\frac{\partial f}{\partial y} = x(x+y)e^{xy} + e^{xy} = e^{xy}(x^2 + xy + 1).$$

Therefore, the gradient of f is given by

$$\vec{\nabla} f =< \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} >=< e^{xy}(xy + y^2 + 1), e^{xy}(x^2 + xy + 1) > .$$

(b) Find the directional derivative of f in the direction of $\vec{v} =< 2, -1 >$ at the point $(2, 2)$.

Solution: We have a nice formula to find the directional derivative in the direction of a unit vector at a point, so we first need to find the unit vector in the direction of $\vec{v}$. To do this, we divide the components of $\vec{v}$ by the length of $\vec{v}$ which is $\sqrt{2^2 + (-1)^2} = \sqrt{5}$. So the unit vector in the direction of $\vec{v}$ is $\vec{u} =< 2/\sqrt{5}, -1/\sqrt{5} >$.

Now the formula for the directional derivative tells us that the directional derivative of f at $P(2, 2)$ in the direction of $\vec{u}$ is

$$\begin{aligned} D_{\vec{u}} f(2,2) &= \vec{\nabla} f(2,2) \cdot \vec{u} \\ &= < 9e^4, 9e^4 > \cdot < 2/\sqrt{5}, -1/\sqrt{5} > \\ &= \frac{18e^4}{\sqrt{5}} - \frac{9e^4}{\sqrt{5}} = \frac{9e^4}{\sqrt{5}}. \end{aligned}$$

(c) Find the maximum rate of change of f at the point $(1, 3)$ and the direction in which it occurs.

Solution: The directional derivative measures the rate of change in a given direction. We know that the directional derivative will always be greatest in the direction of the gradient. In fact, the maximal value of any directional derivative at a point is the length of the gradient at that point. Therefore, we need to find the direction of the gradient at $P(1, 3)$ and the length of the gradient at $P(1, 3)$.

We already calculated the gradient in general, so we just need to plug in the point to get the gradient at that point. So $\vec{\nabla} f(1, 3) =< 13e^3, 5e^3 >$, and so the length of $\vec{\nabla} f(1, 3)$ is

$$\sqrt{(13e^3)^2 + (5e^3)^2} = \sqrt{e^6(169 + 25)} = e^3\sqrt{194}.$$

Altogether, this gives us that the maximum rate of change of f at $P(1, 3)$ is $e^3\sqrt{194}$ in the direction of $< 13/\sqrt{194}, 5/\sqrt{194} >$.

4. Consider the function

$$F(x, y, z) = \sqrt{xyz}$$

and the level surface of that function given by the equation $F(x, y, z) = 6$.

(a) Find the equation of the tangent plane to the level surface at the point $(2, 6, 3)$.

Solution: We need to find the equation of the tangent plane to the level surface. Remember that the gradient of f will be perpendicular to the level surface at every point. Therefore, the normal direction for the tangent plane at the point $(2, 6, 3)$ will be given by the gradient of F at $(2, 6, 3)$. So to write down the equation of the plane, we need to find $\vec{\nabla} F(2, 6, 3)$.

$$\frac{\partial F}{\partial x} = \frac{yz}{2\sqrt{xyz}}, \quad \frac{\partial F}{\partial y} = \frac{xz}{2\sqrt{xyz}}, \quad \text{and} \quad \frac{\partial F}{\partial z} = \frac{xy}{2\sqrt{xyz}}.$$

Therefore, the gradient of F is given by

$$\vec{\nabla} F =< \frac{yz}{2\sqrt{xyz}}, \frac{xz}{2\sqrt{xyz}}, \frac{xy}{2\sqrt{xyz}} >,$$

and so

$$\vec{\nabla} F(2, 6, 3) =< \frac{18}{2\sqrt{36}}, \frac{6}{2\sqrt{36}}, \frac{12}{2\sqrt{36}} >=< \frac{3}{2}, \frac{1}{2}, 1 > .$$

So the equation of the tangent plane to the level surface $F(x,y,z)=6$ at the point $(2,6,3)$ is just the equation of the plane through $(2,6,3)$ perpendicular to $< 3/2, 1/2, 1 >$ which is

$$\frac{3}{2}(x-2)+\frac{1}{2}(y-6)+1(z-3)=0,$$

or

$$3x+y+2z-18=0.$$

(b) Give the parametric equations for the normal line to the level surface at the point $(3,2,6)$.

Solution: The normal line to the level surfave at the point $(3,2,6)$ is just the line through $3,2,6)$ in the direction of the gradient at $(3,2,6)$. So we just need to find the gradient at $(3,2,6)$ which means plugging in the values into the formula we have for the gradient from the previous problem.

$$\vec{\nabla}F(3,2,6) =< \frac{12}{2\sqrt{36}}, \frac{18}{2\sqrt{36}}, \frac{6}{2\sqrt{36}} >=< 1, \frac{3}{2}, \frac{1}{2} > .$$

Therefore, the parametric equations for the normal line to the level surface at the point $(3,2,6)$ are

$$\begin{aligned} x &= 3+t \\ y &= 2+\frac{3}{2}t \\ z &= 6+\frac{1}{2}t \end{aligned}$$

5. Show that $f(x,y)=18x^2-12xy+2y^2+5$ has an infinite number of critical points. Can you say if the critical points are local maxima or minima? Explain.

Solution: To find the critical points of f, we need to find the partial derivatives.

$$\frac{\partial f}{\partial x}=36x-12y \quad \text{and} \quad \frac{\partial f}{\partial y}=-12x+4y.$$

The critical points are all points where both partials are equal to zero or one of them is undefined. Notice that both partial derivatives are defined at all points, so we only need to look for points where both partials are equal to zero.

Solving the system of equations

$$\begin{aligned} 36x-12y &= 0 \\ -12x+4y &= 0 \end{aligned}$$

we can see that there are an infinite number of solutions. For any real number t, the point $(t,3t)$ is a solution and therefore a critical point for the function. Therefore, we do have an infinite number of critical points.

Now we need to decide if the points are local maxima or local minima. Using the second derivative test here won't work here. If we calculate $D(t,3t)$ we see that $D=0$ which doesn't give us any information. So we need to try something else. Notice that $f(t,3t)=5$ for any value of t. Also, we can see that $f(1,0)=23$, so we know that the critical points can't be maxima. If we plug in some more points, we may guess that the critical points are going to be minima. However, without more information, we can't say for sure.

If you use a little ingenuity, you can prove that the critical points are all minima, but I'll let you try to work out how.

6. Find all the local maximum and minimum values and saddle points of the following functions.

(a) $f(x,y)=xy+\frac{1}{x}+\frac{1}{y}$.

Solution: The first step is to find all of the critical points of the function.

$$\frac{\partial f}{\partial x}=y-\frac{1}{x^2} \quad \text{and} \quad \frac{\partial f}{\partial y}=x-\frac{1}{y^2}.$$

Notice that at least one of the partial derivatives will be undefined whenever either $x = 0$ or $y = 0$. However, the fuunction is also undefined then, so we don't need to consider those points.
So we need to consider points which satisfy the following system of equations

$$\begin{aligned} y - \frac{1}{x^2} &= 0 \\ x - \frac{1}{y^2} &= 0. \end{aligned}$$

That means, we need to have $x = 1/y^2$ and $y = 1/x^2$. Solving this, we see that the only critical point will be $(1, 1)$. Now we need to decide if this point is a local maxima, a local minima, or a saddle point.
To use the second derivative test, we need to calculate

$$D(x, y) = \left(\frac{2}{x^3}\right)\left(\frac{2}{y^3}\right) - (1)^2 = \frac{4}{x^3y^3} - 1,$$

and

$$\frac{\partial^2 f}{\partial x^2} = \frac{2}{x^3}.$$

We have $D(1, 1) = 4-1 = 3 > 0$, and so we need to look at $f_{xx}(1, 1) = 2 > 0$. Therefore, the critical point is a local minimum. Since it is the only critical point, there are no other local minima, local maxima, or saddle points.

(b) $f(x, y) = \dfrac{xy}{x + y}$
Solution: Again, we need to find the critical points.

$$\frac{\partial f}{\partial x} = \frac{y}{x + y} - \frac{xy}{(x + y)^2} = \frac{y^2}{(x + y)^2},$$

and

$$\frac{\partial f}{\partial y} = \frac{x}{x + y} - \frac{xy}{(x + y)^2} = \frac{x^2}{(x + y)^2}.$$

Now we need find the points where both partial derivatives equal zero and any points where one of the partials is undefined. Notice that whenever $x = -y$, both partials are undefined. However, the function is also undefined there. So we don't need to consider those points.
The partial derivatives are never both equal to zero, since the only possible point is $(0, 0)$ and the partial derivatives are both undefined there. Therefore, there are no critical points. This means that the function has no local maxima, local minima, or saddle points.

(c) $f(x, y) = e^y(y^2 - x^2)$
*Solution:*Just as in the previous two problems, we need to find the critical points.

$$\frac{\partial f}{\partial x} = -2xe^y \quad \text{and} \quad \frac{\partial f}{\partial y} = e^y(y^2 - x^2) + 2ye^y.$$

Notice that both partial derivatives are defined for all points. Therefore, the only critical points are going to be places where both partial derivatives are equal to zero. So we need to solve the following system of two equations

$$\begin{aligned} -2xe^y &= 0 \\ e^y(y^2 - x^2) + 2ye^y &= 0, \end{aligned}$$

The first equation will only be satisfied if $x = 0$. Putting that into the second equation, we see that we have $y = 0$ or $y = -2$. Therefore, there are two critical points: $(0, 0)$ and $(0, -2)$.
To determine what the critical points are, we can try the second derivative test. First, we calculate

$$D(x, y) = (-2e^y)\left[e^y(y^2 - x^2) + 2ye^y + 2e^y + 2ye^y\right] - (-2xe^y)^2 = -2y^2e^{2y} + 2x^2e^{2y} - 4e^{2y} - 8ye^{2y} - 4x^2e^{2y},$$

and

$$\frac{\partial^2 f}{\partial x^2} = -2e^y.$$

For the critical point $(0,0)$, we have $D(0,0) = -4$. This means that we don't have to check the value of the second partial; the point $(0,0)$ is saddle point. For the critical point $(0,-2)$, we have $D(0,-2) = 4e^{-4} > 0$. So we need to calculate the value $f_{xx}(0,-2) = -2e^{-4} < 0$. Therefore, the point $(0,-2)$ is a local maximum.

Since there aren't any other critical points, there aren't any other local maxima, local minima, or saddle points.