Homework 6 Solutions, Calc III

1. Find the absolute maximum and absolute minimum values of the function f on the domain D.

   (a) $f(x,y) = x^2 + y^2 + x^2y + 4$ and $D = \{\, (x,y) \mid |x| \le 1, |y| \le 1 \,\}$.
   Solution: There are two steps we have to go through to solve this problem. The first step is to fint the critical points of the function which are in the domain. So we need to consider the first derivatives

   $$\frac{\partial f}{\partial x} = 2x + 2xy \quad \text{and} \quad \frac{\partial f}{\partial y} = 2y + x^2.$$

   Setting both the derivatives to zero, we need to solve the system

   $$\begin{aligned} 2x(1+y) &= 0 \\ 2y + x^2 &= 0 \end{aligned}$$

   The first equation gives that $x = 0$ or $y = -1$. If $x = 0$, then $y = 0$ from the second equation. If $y = -1$, then $x = \pm\sqrt{2}$ from the second equation. So the critical points are $(0,0)$, $(\sqrt{2}, -1)$, and $(-\sqrt{2}, -1)$. Simple calculation shows $f(0,0) = 4$, $f(\sqrt{2}, -1) = 6$ and $f(-\sqrt{2}, -1) = 6$.

   The next step is to find the maximum and minimum values of the function on the boundary. The boundary has 4 pieces:

   - The piece where $x = -1$ and $-1 \le y \le 1$.
     On this piece of the boundary, we have $f(1,y) = g(y) = y^2 + y + 5$ and we need to maximize and minimize this function for $-1 \le y \le 1$. $g'(y) = 2y + 1$, so the critical point is $y = -1/2$. The second derivative test tells us that $y = -1/2$ is a minimum. So the minimum value of the function on this piece of the boundary is $f(-1, -1/2) = 19/4$. The maximum value will either be at $y = 1$ or $y = -1$. It is easy to see that the maximum value of f on this piece of the boundary will is $f(-1,1) = 7$.
   - The piece where $x = 1$ and $-1 \le y \le 1$.
     On this piece of the boundary we have $f(1,y) = g(y) = y^2 + y + 5$ and we need to maximize and minimize this function for $-1 \le y \le 1$. Notice that we already did this for the previous piece of the boundary. Therefore, the maximum value on this piece of the boundary is $f(1,1) = 7$ and the minimum value is $f(1, -1/2) = 19/4$.
   - The piece where $y = -1$ and $-1 \le x \le 1$.
     On this piece of the boundary we have $f(x,-1) = h(x) = 5$ and we need to maximize and minimize this function for $-1 \le x \le 1$. Since this is a constant function. We have that the minimum value of f on this piece of the boundary is 5, and the maximum value is 5.
   - The piece where $y = 1$ and $-1 \le x \le 1$.
     On this piece of the boundary we have $f(x,1) = h(x) = 2x^2 + 5$ and we need to maximize and minimize this for $-1 \le x \le 1$. $h'(x) = 2x$ so the critical point is $x = 0$ and the second derivative test tells us that this will be a minimum. So the minimum value of the function on this piece of the boundary is $f(0,1) = 5$. The maximum will either come at $x = 1$ or $x = -1$. It is easy to see that the maximum value of the function on this piece of the boundary will come at both $(1,1)$ and $(-1,1)$. So the maximum value of the function on this piece of the boundary is $f(1,1) = f(-1,1) = 7$.

   Putting all this together, we see that the absolute maximum value of the function on the region is 7 which happens at $(-1,1)$ and $(1,1)$, and the minimum value of the function on the region is 4 which happens at $(0,0)$.

   (b) $f(x,y) = xy^2$ and $D = \{\, (x,y) \mid x \ge 0, y \ge 0, x^2 + y^2 \le 3 \,\}$.
   Solution: Just as in problem (a), there are two steps. The first is to find the critical points. The partial derivatives are

   $$\frac{\partial f}{\partial x} = y^2 \quad \text{and} \quad \frac{\partial f}{\partial y} = 2xy.$$

   Setting them equal to zero, we see that the critical points are all points of the form $(x,0)$ in the domain. Notice that at any point $(x,0)$, $f(x,0) = 0$.

   The second step is to consider what happens on the boundary. The boundary has 3 pieces:

   - The piece where $x = 0$.
     On this piece, the function becomes $f(0,y) = 0$ which is a constant. So on this piece of the boundary, the maximum value of the function is 0 and the minimum value of the function is 0.

- The piece where $y = 0$.
On this ouece, the function becomes $f(x, 0) = 0$ which is a constant. So again, on this piece of the boundary, the maximum value of the function is 0 and the minimum value of the function is 0.
- The piece where $x^2 + y^2 = 3$.
On this piece of the boundary, we want to find the extreme values of the function with the constraint $x^2 + y^2 = 3$. So we have a Lagrange multiplier problem. We need to solve the following system of equations

$$\begin{aligned} y^2 &= \lambda 2x & (1) \\ 2xy &= \lambda 2y & (2) \\ x^2 + y^2 &= 3 & (3) \end{aligned}$$

Equation (2) gives us that either $\lambda = x$ or $y = 0$. If $y = 0$, equation (3) tells us that $x = \pm\sqrt{3}$. So we get the points $(\pm\sqrt{3}, 0)$, and at these points, we have $f(\sqrt{3}, 0) = 0 = f(-\sqrt{3}, 0)$.
If $\lambda = x$, we get $2x^2 = y^2$ from equation (1). Then plugging into equation (3), gives $3x^2 = 3$ or $x^2 = 1$. Solving this gives $x = \pm 1$, however, remember that the domain we're looking at has $x \geq 0$. So we only need to consider $x = 1$. If $x = 1$, we use our equation $2x^2 = y^2$ to get $y = \pm\sqrt{2}$. Again we have $y \geq 0$ in our domain, so we only need to worry about $y = \sqrt{2}$. So we have the point $(1, \sqrt{2})$ and $f(1, \sqrt{2}) = 2$.
Taking all of this into account, we have that the maximum value of f on this piece of the boundary is 2 and the minimum value of f on this piece of the boundary is 0.

Combining the reaults from both steps we get that the absolute maximum value of f on the domain is 2 which happens at $(1, \sqrt{2})$. The absolute minimum value of the function on the domain is 0 which happens at all points of the fomr $(x, 0)$ or $(0, y)$ in the domain.

(c) $f(x, y) = \sin(xy)\cos(xy)$ and $D = \{\,(x, y)\,|\, 0 \leq x \leq \pi/2, 0 \leq y \leq \pi\,\}$.
Solution: We have to do the two steps again. First find the critical points. Consider the first partial derivatives

$$\frac{\partial f}{\partial x} = y(\cos^2(xy) - \sin^2(xy)) = y(\cos(xy) - \sin(xy))(\cos(xy) + \sin(xy)),$$

and

$$\frac{\partial f}{\partial y} = x(\cos^2(xy) - \sin^2(xy)) = x(\cos(xy) - \sin(xy))(\cos(xy) + \sin(xy)).$$

Setting them both equal to zero, we see that the critical points are going to be $(0, 0)$ and any points where $\cos(xy) = \sin(xy)$ or $\cos(xy) = -\sin(xy)$. This gives us the point $(0, 0)$, all the points where $xy = \pi/4$, all the points where $xy = 3\pi/4$, all the points where $xy = 5\pi/4$, all the points where $xy = 7\pi/4$, etc. However, notice that in our domain $x \geq 0$ and $y \geq 0$, so we only care about the poits where xy is positive (that's why I didn't include the negative angles in the list). Also, you can check, but there aren't any points in the domain with $xy \geq 7\pi/4$. Therefore, we only have the cases $xy = \pi/4$, $xy = 3\pi/4$, and $xy = 5\pi/4$.
If we caluculate the value of the function at all of the critical points, we get the following:

- $f(0, 0) = 0$
- If $xy = \pi/4$, then $f(x, y) = 1/2$.
- If $xy = 3\pi/4$, then $f(x, y) = -1/2$.
- If $xy = 5\pi/4$, then $f(x, y) = 1/2$.

Secondly we need to consider what happens on the boundary. The boundary has 4 pieces:

- The piece where $x = 0$.
On this piece, the function becomes $f(0, y) = 0$ which is a conatant. Therefore, the maximum value of the funciton on this piece of the boundary is 0, and the minimum value is 0. They happen at all the points.
- The piece where $y = 0$.
On this piece, the function becomes $f(x, 0) = 0$ which is a constant. Therefore the maximum value of the function on this piece of the boundary is 0, and the minimum value is 0.
- The piece of the boundary where $x = \pi/2$.
On this piece of the boundary, the function becomes $f(\pi/2, y) = g(y) = \cos(\pi y/2)\sin(\pi y/2)$. We need to maximize and minimize $g(y)$ for $0 \leq y \leq \pi$. If you do this you can see that the critical points of $g(y)$ are $y = 1/2, 3/2, 5/2$. Plugging in the points we get $f(\pi/2, 1/2) = 1/2 = f(\pi/2, 5/2)$ and $f(\pi/2, 3/2) = -1/2$. So the maximum value of the function on this piece of the boundary is 1/2 at the

points $(\pi/2, 1/2)$ and $(\pi/2, 5/2)$. The minimum value of this function on this piece of the boundary is $-1/2$ at the point $(\pi/2, 3/2)$.

- The piece of the boundary where $y = \pi$.
  On this piece of the boundary, the function becomes $f(x, \pi) = h(x) = \cos(\pi x)\sin(\pi x)$. We need to maximize and minimize $h(x)$ for $0 \le x \le \pi/2$. If you do this you can see that the critical points of $h(x)$ are $x = 1/4, 3/4, 5/4$. Plugging in the points we get $f(1/4, \pi) = 1/2 = f(5/4, \pi)$ and $f(3/4, \pi) = -1/2$. So the maximum value of the function on this piece of the boundary is $1/2$ at the points $(1/4, \pi)$ and $(5/4, \pi)$. The minimum value of this function on this piece of the boundary is $-1/2$ at the point $(3/4, \pi)$.

Putting all of this together, we see that the absolute maximum value of the function on the domain is $1/2$ which happens at all the points with $xy = \pi/4$ or $xy = 5\pi/4$, The absolute minimum value of the function on the domain is $-1/2$ which happens at all the points with $xy = 3\pi/4$.

2. Find the maximum and minimum volumes of a rectangular box whose surface area is $1500cm^2$ and total edge length is $200cm$.

   Solution: Here we can solve this problem by setting up a Lagrange multiplier problem. If we let x be the length of the box, y be the width of the box and z be the height of the box, the volume of the box will be xyz. So we want to maximize and minimize the function $f(x, y, z) = xyz$ under the conditions $2xy + 2xz + 2yz = 1500$ and $4x + 4y + 4z = 200$. We can simplify the two conditions by dividing both sides of the first equation by 2 and both sides of the second condition by 4.

   Using Lagrange multipliers, we get the following system of equations:

$$\begin{aligned} yz &= \lambda(y+z) + \mu & (4) \\ xz &= \lambda(x+z) + \mu & (5) \\ xy &= \lambda(x+y) + \mu & (6) \\ xy + xz + yz &= 750 & (7) \\ x + y + z &= 50 & (8) \end{aligned}$$

   We need to find all of the solutions to this system. If we multiply equation (4) by x and equation (5) by y, we obtain

$$\begin{aligned} xyz &= \lambda x(y+z) + \mu x \\ xyz &= \lambda y(x+z) + \mu y. \end{aligned}$$

   This gives $\lambda x(y+z) + \mu x = \lambda y(x+z) + \mu y$ which simplifies to

$$x(\lambda z + \mu) = y(\lambda z + \mu).$$

   This equation gives us two possibilities. Either $x = y$ or $\lambda z + \mu = 0$.

   - If $x = y$, we can use equation (8) to get $2x + z = 50$. This gives $z = 50 - 2x$. Plugging this in to equation (7), we get $3x^2 - 100x - 750 = 0$. Solving this for x, we get

$$x = \frac{50 \pm 5\sqrt{10}}{3}$$

     which gives us two points using $x = y$ and $z = 50 - 2x$:

$$\left(\frac{50+5\sqrt{10}}{3}, \frac{50+5\sqrt{10}}{3}, \frac{50-5\sqrt{10}}{3}\right) \text{ and } \left(\frac{50-5\sqrt{10}}{3}, \frac{50-5\sqrt{10}}{3}, \frac{50+5\sqrt{10}}{3}\right).$$

     At these two points, the value of f is

$$250\left(\frac{50+5\sqrt{10}}{3}\right) \text{ and } 250\left(\frac{50-5\sqrt{10}}{3}\right)$$

     respectively.

   - If $\lambda z + \mu = 0$, then multiply both sides of equation (5) by y and multiply both sides of equation (6) by z. This gives us

$$y(\lambda x + \mu) = z(\lambda x + \mu).$$

     This equation gives us two cases, either $y = z$ or$\lambda x + \mu = 0$.

– If $y = z$, we can solve for y the same way we solved for x above. This gives us the points

$$\left(\frac{50-5\sqrt{10}}{3}, \frac{50+5\sqrt{10}}{3}, \frac{50+5\sqrt{10}}{3}\right) \text{ and } \left(\frac{50+5\sqrt{10}}{3}, \frac{50-5\sqrt{10}}{3}, \frac{50-5\sqrt{10}}{3}\right).$$

At these two points, the value of f is

$$250\left(\frac{50+5\sqrt{10}}{3}\right) \text{ and } 250\left(\frac{50-5\sqrt{10}}{3}\right)$$

respectively.

– If $\lambda x + \mu = 0$, then we have $\lambda z + \mu = \lambda x + \mu$. This simplifies to $\lambda x = \lambda z$. This gives us either $\lambda = 0$ or $x = z$.

  * If $x = z$, then we can solve for x as above. This gives us the points

$$\left(\frac{50-5\sqrt{10}}{3}, \frac{50+5\sqrt{10}}{3}, \frac{50+5\sqrt{10}}{3}\right) \text{ and } \left(\frac{50+5\sqrt{10}}{3}, \frac{50-5\sqrt{10}}{3}, \frac{50-5\sqrt{10}}{3}\right).$$

  At these two points, the value of f is

$$250\left(\frac{50+5\sqrt{10}}{3}\right) \text{ and } 250\left(\frac{50-5\sqrt{10}}{3}\right)$$

  respectively.

  * If $\lambda = 0$, then the equation $\lambda x + \mu = 0$ tells us that $\mu = 0$. Therefore equations (4), (5) and (6) from our system turn into $yz = 0$, $xz = 0$ and $xy = 0$. This gives three possibilities either $x = 0$ and $y = 0$, $x = 0$ and $z = 0$, or $y = 0$ and $z = 0$. Notice that all of these cases violate equation (7) from our system. Therefore, they don't give us any solutions.

Putting all of this together, we can see that the absolute maximum value of the function (that is the maximum volume of the box in cm^3) is

$$250\left(\frac{50+5\sqrt{10}}{3}\right)$$

which happens at three points. The absolute minimum value of the function (or the minimum volume of the box in cm^3) is

$$250\left(\frac{50-5\sqrt{10}}{3}\right)$$

which happens at three points.

3. Find the extreme values of f on the region described by the inequality.

$$f(x, y) = 2x^2 + 3y^2 - 4x - 5, \quad x^2 + y^2 \leq 16$$

Solution: For this problem, we need to follow two steps, just as in the problem 1. First we need to find the critical points of the function. The first partials are

$$\frac{\partial f}{\partial x} = 4x - 4 \quad \text{and} \quad \frac{\partial f}{\partial y} = 6y.$$

That means the only critical point is $(1, 0)$. The value of f at this point is $f(1, 0) = -7$.

Next we need to consider what happens on the boundary. Since the boundary is defined by the equation $x^2 + y^2 = 16$, this leads to a Lagrange multiplier problem. The system of equations we need to solve here is

$$\begin{aligned} 4x - 4 &= \lambda 2x & (9) \\ 6y &= \lambda 2y & (10) \\ x^2 + y^2 &= 16 & (11) \end{aligned}$$

Looking at equation (10), we get that either $y = 0$ or $\lambda = 3$.

- If $y = 0$, equation (11) gives us that $x = \pm 4$. So we get two points $(4, 0)$ and $(-4, 0)$. The value of f at these two points is $f(4, 0) = 11$ and $f(-4, 0) = 43$.
- If $\lambda = 3$, then equation (9) becomes $4x - 4 = 6x$ which gives $x = -2$. Since $x = -2$, equation (11) gives $y = \pm\sqrt{12}$. This gives us two points $(-2\sqrt{12})$ and $(-2, -\sqrt{12})$. The value of f at these points is $f(-2, \sqrt{12}) = 47 = f(-2, -\sqrt{12})$.

Putting all of this together, we get that the absolute maximum value of the funciton on the domain is 47 which happens at the points $(-2, \sqrt{12})$ and $(-2, -\sqrt{12})$. The absolute minimum value of the function on the domain is -7 which happens at the point $(1, 0)$.

4. Use Lagrange multipliers to find the maximum and minimum values of the function subject to the given constraint(s).

(a) $f(x, y) = e^{xy} \quad x^3 + y^3 = 16$.

Solution: Lagrange multipliers sets up the following system of equations

$$
\begin{aligned}
ye^{xy} &= \lambda 3x^2 && (12)\\
xe^{xy} &= \lambda 3y^2 && (13)\\
x^3 + y^3 &= 16 && (14)
\end{aligned}
$$

If we multiply both sides of equation (12) by x and both sides of equation (13) by y, we get $xye^{xy} = \lambda 3x^3$ and $xye^{xy} = \lambda 3y^3$. This gives $lambda 3x^3 = \lambda 3y^3$. Therefore, either $\lambda = 0$ or $x^3 = y^3$.

- If $\lambda = 0$, the equation (12) and equation (13) give us that $x = 0$ and $y = 0$ which violates equation (14). So this doesn't give us anything.
- If $x^3 = y^3$, then $x = y$. Plugging this into equation (14), we get $2x^3 = 16$. This simplifies to $x = 2$, so we get the point $(2, 2)$. The value of the funciton at this point is $f(2, 2) = e^4$. This is the maximum value of the funciton given the constraint. You can tell since the point $(0, \sqrt[3]{16})$ is on the surface $x^3 + y^3 = 16$, and $f(0, \sqrt[3]{16}) = e^0 = 1$. So this funciton doesn't have a minimum given the constraint.

(b) $f(x, y) = x^3 + y^3 \quad x^2 + 4y^2 = 65$.

Solution: In this case we get the system of equations

$$
\begin{aligned}
3x^2 &= \lambda 2x && (15)\\
3y^2 &= \lambda 8y && (16)\\
x^2 + 4y^2 &= 65 && (17)
\end{aligned}
$$

Equation (15) gives either $x = 0$ or $\lambda = 3x/2$.

- If $x = 0$, equation (17) gives $y = \pm\sqrt{65}/2$. So we have two points $(0, \sqrt{65}/2)$ and $(0, -\sqrt{65}/2)$. The value of the funciton at these points is $f(0, \sqrt{65}/2) = 65\sqrt{65}/8$, and $f(0, -\sqrt{65}/2) = -65\sqrt{65}/8$.
- If $\lambda = 3x/2$, we can use equation (16) to see that either $y = 0$ or $\lambda = 3y/8$.
  - If $y = 0$, then equation (17) gives us $x = \pm\sqrt{65}$. So we get two points $(\sqrt{65}, 0)$ and $(-\sqrt{65}, 0)$. The value of f at these two points is $f(\sqrt{65}, 0) = 65\sqrt{65}$ and $f(-\sqrt{65}, 0) = -65\sqrt{65}$.
  - If $\lambda = 3y/8$, then we have $3x/2 = 3y/8$. This gives $4x = y$. Plugging this into equation (17), we have $x = \pm\sqrt{65/17}$. This gives us two points $(\sqrt{65/17}, 4\sqrt{65/17})$ and $(-\sqrt{65/17}, -4\sqrt{65/17})$. The value of the f at the points is

$$
f(\sqrt{65/17}, 4\sqrt{65/17}) = \frac{4225}{17}\sqrt{\frac{65}{17}} \text{ and } f(\sqrt{65/17}, 4\sqrt{65/17}) = -\frac{4225}{17}\sqrt{\frac{65}{17}}
$$

.

Putting all of this together, we can see that the maximum value of the function given the constraint is $65\sqrt{65}$ and the absolute minimum value of the funciton is $-65\sqrt{65}$.

(c) $f(x, y, z) = x^2y^2z^2 \quad x^2 + y^2 + z^2 = 1$.

Solution: This problem gives us the following system of equations.

$$
\begin{aligned}
2xy^2z^2 &= \lambda 2x && (18)\\
2x^2yz^2 &= \lambda 2y && (19)\\
2x^2y^2z &= \lambda 2x && (20)\\
x^2 + y^2 + z^2 &= 1 && (21)
\end{aligned}
$$

If we multiply both sides of equation (18) by x and both sides of equation (19) by y and both sides of equation (20) by z, we get

$$\begin{aligned} 2x^2y^2z^2 &= \lambda 2x^2 \\ 2x^2y^2z^2 &= \lambda 2y^2 \\ 2x^2y^2z^2 &= \lambda 2z^2 \end{aligned}$$

So this gives us $\lambda 2x^2 = \lambda 2y^2 = \lambda 2z^2$. This leads to two possibilities. Either $\lambda = 0$ or $x^2 = y^2 = z^2$.

- If $\lambda = 0$, the equations (18), (19), and (20) give us that either $x = 0$, $y = 0$ or $z = 0$. Notice that in any of these cases, the value of f will be 0.
- If $x^2 = y^2 = z^2$, then equation (21) gives $x = \pm 1/\sqrt{3}$. Altogether, this gives us eight points since we also have $y = \pm 1/\sqrt{3}$ and $z = \pm 1/\sqrt{3}$. At any of these points the value of the f will be $1/27$.

Putting all of this together, we can see that the maximum value of f with this constraint is $1/27$ and the minimum value of f with this restraint is 0.

(d) $f(x, y, z) = xy + yz \quad xy = 1 \quad y^2 + z^2 = 1$.
Solution: For this problem, we get the following system of equations

$$\begin{aligned} y &= \lambda y && (22) \\ x + z &= \lambda x + \mu 2y && (23) \\ y &= \mu 2z && (24) \\ xy &= 1 && (25) \\ y^2 + z^2 &= 1 && (26) \end{aligned}$$

Equation (22) tells us that either $y = 0$ or $\lambda = 1$. However, equation (25) tells us that it is impossible to have $y = 0$. Therefore, $\lambda = 1$. So equation (23) simplifies to $z = \mu 2y$. Combine this with equation (24) and we have $y = 4\mu^2 y$. Since we know that $y \neq 0$, this means that $\mu = 1/2$. Therefore, we have $z = y$. So equation (26) we get $2y^2 = 1$. Therefore, $y = \pm\sqrt{1/2}$. Equation (25) then gives $x = \pm\sqrt{2}$. So we get two points $(\sqrt{2}, \sqrt{1/2}, \sqrt{1/2})$ and $(-\sqrt{2}, -\sqrt{1/2}, -\sqrt{1/2})$. The value of f at these points is

$$f(\sqrt{2}, \sqrt{1/2}, \sqrt{1/2}) = 3/2 \quad \text{and} \quad f(-\sqrt{2}, -\sqrt{1/2}, -\sqrt{1/2}) = 3/2.$$

This is the maximum value that f can take given the two constraints. The way you can tell this is to notice that the point $(1, 1, 0)$ satisfies both constraints, and $f(1, 1, 0) = 1$.