Project, Number Systems
Friday December 12th, 2008

1 Introduction

In this project you will explore two different ways of defining the real numbers from the rationals, and you will prove that they are equivalent. In other words, they both give the same real number system that we are used to. The two constructions involve **Dedekind cuts** and **completion**. In both projects, we will assume that we have defined the rational numbers $\mathbb{Q}$. We can think of them as the set

$$\mathbb{Q} = \left\{ \frac{m}{n} \;\middle|\; m \in \mathbb{Z}, n \in \mathbb{N} \right\},$$

and we know that they satisfy the following properties:

Axiom 1.1. *There is a set $\mathbb{Q}$ with two binary operations addition and multiplication which satisfies the following properties for any $p, q, r \in \mathbb{Q}$:*

1. $p + q = q + p$.
2. $p + (q + r) = (p + q) + r$.
3. $p(q + r) = pq + pr$.
4. $pq = qp$
5. $p(qr) = (pq)r$

Axiom 1.2. *There is an element $0 \in \mathbb{Q}$ so that for any $q \in \mathbb{Q}$, $q + 0 = q$*

Axiom 1.3. *There is an element $1 \in \mathbb{Q}$ with $1 \neq 0$ so that for every $q \in \mathbb{Q}$, $q \cdot 1 = q$.*

Axiom 1.4. *For any element $p \in \mathbb{Q}$ there is an element $-p \in \mathbb{Q}$ such that $p + (-p) = 0$.*

Axiom 1.5. *For any element $q \in \mathbb{Q} - \{0\}$, there is an element $q^{-1} \in \mathbb{Q}$ such that $q \cdot q^{-1} = q^{-1} \cdot q = 1$.*

We can define the set $\mathbb{Q}_{>0}$ to be the set

$$\mathbb{Q}_{>0} = \{ \frac{m}{n} \,|\, m \in \mathbb{N}, n \in \mathbb{N} \}.$$

We know that $\mathbb{Q}_{>0}$ satisfies the following properties:

- If $p, q \in \mathbb{Q}_{>0}$, then $pq \in \mathbb{Q}_{>0}$.
- If $p, q \in \mathbb{Q}_{>0}$, then $p + q \in \mathbb{Q}_{>0}$.
- For any $p \in \mathbb{Q}$, exactly one of the following is true: $p = 0$, $p \in \mathbb{Q}_{>0}$, or $-p \in \mathbb{Q}_{>0}$.

Using this we can define for $p, q \in \mathbb{Q}$, $p < q$ if $q - p \in \mathbb{Q}_{>0}$. This leads to one last property of $\mathbb{Q}$ that we will need. In fact, we can take it as an axiom.

Axiom 1.6. *For any $q \in \mathbb{Q}$, there exists an $n \in \mathbb{N}$ such that $q < n$.*

2 Dedekind Cuts

The purpose of Dedekind cuts is to provide a sound logical foundation for the real number system assuming that you have constructed the rational numbers. Dedekind's motivation behind this project is to notice that a real number α, intuitively, is completely determined by the rationals strictly smaller than α and those strictly larger than α. Concerning the completeness or continuity of the real line, Dedekind notes in [2] that

> If all points of the straight line fall into two classes such that every point of the first class lies to the left of every point of the second class, then there exists one and only one point which produces this division of all points into two classes, this severing of the straight line into two portions.

Dedekind defines a point to produce the division of the real line if this point is either the least or greatest element of either one of the classes mentioned above. He further notes that the completeness property, as he just phrased it, is deficient in the rationals, which motivates the definition of reals as cuts of rationals. Because all rationals greater than α are really just excess baggage, we prefer to sway somewhat from Dedekind's original definition. Instead, we adopt the following definition.

Definition 2.1. *A* Dedekind cut *is a subset α of the rational numbers $\mathbb{Q}$ that satisfies these properties:*

1. *α is not empty.*
2. *$\mathbb{Q} - \alpha$ is not empty. Here $\mathbb{Q} - \alpha$ means the complement of α in $\mathbb{Q}$.*
3. *α contains no greatest element*
4. *For $x, y \in \mathbb{Q}$, if $x \in \alpha$ and $y < x$, then $y \in \alpha$ as well.*

Dedekind cuts are particularly appealing for two reasons. First, they make it very easy to prove the completeness, or continuity of the real line. Also, they make it quite plain to distinguish the rationals from the irrationals on the real line, and put the latter on a firm logical foundation. In the construction of the real numbers from Dedekind cuts, we make the following definition:

Definition 2.2. *A* real number *is a Dedekind cut. We denote the set of all real numbers by $\mathbb{R}$ and we order them by set-theoretic inclusion, that is to say, for any $\alpha, \beta \in \mathbb{R}$,*

$$\alpha < \beta \text{ if and only if } \alpha \subset \beta$$

where the inclusion is strict. We further define $\alpha = \beta$ as real numbers if α and β are equal as sets. As usual, we write $\alpha \leq \beta$ if $\alpha < \beta$ or $\alpha = \beta$. Moreover, a real number α is said to be irrational *if $\mathbb{Q} - \alpha$ contains no least element.*

With this definition of the real numbers, it is possible to show that all the axioms we had in class for the real numbers hold. To do this we need to make explicit some definitions, specifically what we mean by $\alpha + \beta$, $\alpha\beta$, 1, 0, and α^{-1} for real numbers α and β.

Definition 2.3. *Given two real numbers α and β, we define*

- *The* additive identity*, denoted* 0*, is*
$$0 := \{x \in \mathbb{Q} : x < 0\}$$
- *The* multiplicative identity*, denoted* 1*, is*
$$1 := \{x \in \mathbb{Q} : x < 1\}$$
- Addition *of α and β denoted $\alpha + \beta$ is*
$$\alpha + \beta := \{x + y : x \in \alpha,\ y \in \beta\}$$
- *The* opposite *of α, denoted $-\alpha$, is*
$$-\alpha := \{x \in \mathbb{Q} : -x \notin \alpha, \text{ but } -x \text{ is not the least element of } \mathbb{Q} - \alpha\}$$
- *The* absolute value *of α, denoted $|\alpha|$, is*
$$|\alpha| := \begin{cases} \alpha, & \text{if } \alpha \geq 0 \\ -\alpha, & \text{if } \alpha \leq 0 \end{cases}$$
- *If $\alpha, \beta > 0$, then* multiplication *of α and β, denoted $\alpha \cdot \beta$, is*
$$\alpha \cdot \beta := \{z \in \mathbb{Q} : z \leq 0 \text{ or } z = xy \text{ for some } x \in \alpha,\ y \in \beta \text{ with } x, y > 0\}$$

  In general,
$$\alpha \cdot \beta := \begin{cases} 0, & \text{if } \alpha = 0 \text{ or } \beta = 0 \\ |\alpha| \cdot |\beta| & \text{if } \alpha > 0,\ \beta > 0 \text{ or } \alpha < 0,\ \beta < 0 \\ -(|\alpha| \cdot |\beta|) & \text{if } \alpha > 0,\ \beta < 0 \text{ or } \alpha > 0,\ \beta < 0 \end{cases}$$

- *The* inverse *of* $\alpha > 0$, *denoted* α^{-1}, *is*

$$\alpha^{-1} := \{x \in \mathbb{Q} : x \leq 0 \text{ or } x > 0 \text{ and } (1/x) \notin \alpha, \text{ but } 1/x \text{ is not the least element of } \mathbb{Q} - \alpha\}$$

If $\alpha < 0$,

$$\alpha^{-1} := -(|\alpha|)^{-1}$$

Check that the above definitions mean that $\mathbb{R}$ satisfies all 7 of the axioms we had for the real numbers. You'll have to use the properties of $\mathbb{Q}$ to properly check everything. It is important to point out that in two steps, in showing that inverses and opposites are properly defined, you require the last axiom of $\mathbb{Q}$. The rationals correspond to the Dedekind cuts α for which $\mathbb{Q} - \alpha$ contains a least member.

3 Cauchy sequences

An alternative description of the real numbers is in terms of Caucy sequences of rational numbers. In order to do that, we need to know what Cauchy sequences are.

Definition 3.1. *A sequence* $\{a_n\}$ *of rational numbers is called a* Cauchy sequence *if for any rational* $\epsilon > 0$, *there is an* $N \in \mathbb{N}$ *such that*

$$|a_n - a_m| < \epsilon \text{ whenever } n, m \geq N.$$

In order to prove that we can define the real numbers in terms of Cauchy sequences, you will need to prove some facts about Cauchy sequnces of rational numbers.

- Using the definition of convergence from class and the definition of real numbers from class, prove that any Cauchy sequence of rational numbers converges to a real number. To do this you might want to show that any Cauchy sequence is bounded, and then use that to show that the Cauchy sequence has to converge.

Since you proved in a previous assignment that the limit of a sequence of real numbers is unique, we can think of a Cauchy sequence as determining a real number since any Cauchy sequence has a unique limit. You will have to show that any real number can actually be thought of a the limit of a Cauchy sequence of rational numbers. We start with the following definition.

Definition 3.2. *A real number* α *is a Cauchy sequence of rational numbers, that is* $\alpha = \{a_n\}$. *We will denote the set of all real numbers by* $\mathbb{R}$.

With this definition, addition and multiplication are defined to be the addition and multiplication of sequences. So, for two real numbers $\alpha = \{a_n\}$ and $\beta = \{b_n\}$, we have $\alpha + \beta = \{a_n + b + n\}$ and $\alpha\beta = \{a_n b_n\}$. Similarly, two real numbers $\alpha = \{a_n\}$ and $\beta = \{b_n\}$ are said to be equal if $\{a_n - b_n\}$ converges to 0. You will also need the following definition.

Definition 3.3. *For two real numbers* $\alpha = \{a_n\}$ *and* $\beta = \{b_n\}$, *we say* $\alpha < \beta$ *if there exists* $N \in \mathbb{N}$ *such that*

$$a_n < b_n \text{ whenever } n \geq N.$$

In other words, as long as $\beta - \alpha > 0$.

To make sure that these definitions are good, you need to show the following:

- If $\{a_n\}$ and $\{b_n\}$ are Cauchy sequences, then $\{a_n + b_n\}$ is a Cauchy sequence.
- If $\{a_n\}$ and $\{b_n\}$ are Cauchy sequences, then $\{a_n b_n\}$ is a Cauchy sequence.
- If $\{a_n\}$ is a Cauchy sequence with $\{a_n\} \neq 0$, then $\{\frac{1}{a_n}\}$ is a Cauchy sequence. Here 0 represents the constant sequence $\{0\}$.

After you have proven these facts, use them to show that $\mathbb{R}$ satisfies all seven axioms of the real numbers that we had in class. You will need to explicitly define $-\alpha$ and α^{-1} for a real number α.

References

[1] Courant, Richard and Robbins, Herbert. *What is Mathematics?* pp. 68-72 Oxford University Press, Oxford, 1969

[2] Dedekind, Richard. *Essays on the Theory of Numbers* Dover Publications Inc, New York 1963

[3] Rudin, Walter *Principles of Mathematical Analysis* pp. 17-21 McGraw-Hill Inc, New York, 1976

[4] Spivak, Michael. *Calculus* pp. 569-596 Publish or Perish, Inc. Houston, 1994