# Project Supplement, Number Systems

Friday December 12th, 2008

## 1 Overview

This supplement is an attempt to clarify exactly what you are supposed to do for the project. In class we defined the real numbers in terms of seven axioms which I will list below for convenience.

**Axiom 1.1.** *There is a set $\mathbb{R}$ with two binary operations addition and multiplication which satisfies the following properties for any $x, y, z \in \mathbb{R}$:*

1. $x + y = y + x$.
2. $x + (y + z) = (x + y) + z$.
3. $x(y + z) = xy + xz$.
4. $xy = yx$
5. $x(yz) = (xy)z$

**Axiom 1.2.** *There is an element $0 \in \mathbb{R}$ so that for any $y \in \mathbb{R}$, $y + 0 = y$*

**Axiom 1.3.** *There is an element $1 \in \mathbb{R}$ with $1 \neq 0$ so that for every $y \in \mathbb{R}$, $y \cdot 1 = y$.*

**Axiom 1.4.** *For any element $x \in \mathbb{R}$ there is an element $-x \in \mathbb{R}$ such that $x + (-x) = 0$.*

**Axiom 1.5.** *For any element $y \in \mathbb{R} - \{0\}$, there is an element $y^{-1} \in \mathbb{R}$ such that $y \cdot y^{-1} = y^{-1} \cdot y = 1$.*

**Axiom 1.6.** *There is a subset $\mathbb{R}_{>0} \subset \mathbb{R}$ which satisfies the following properties:*

- *If $x, y \in \mathbb{R}_{>0}$, then $xy \in \mathbb{R}_{>0}$.*
- *If $x, y \in \mathbb{R}_{>0}$, then $x + y \in \mathbb{R}_{>0}$.*
- *For any $x \in \mathbb{R}$, exactly one of the following is true: $x = 0$, $x \in \mathbb{R}_{>0}$, or $-x \in \mathbb{R}_{>0}$.*

**Axiom 1.7.** *Any non-empty subset of $\mathbb{R}$ shich is bounded above has a least upper bound.*

This project is about how we could define the real numbers by constructing them out of the rational numbers. This is different than how we did it in class.

## 2 Dedekind Cuts

For this section, I define a Dedekind cut of the rationals, and then define a real number to be a Dedekind cut. So the real numbers are the collection of Dedekind cuts of the rationals.

1. You need to prove that this definition of the real numbers satisfies the axioms above. Axiom 7 will most likely be the trickiest to prove.

## 3 Cauchy Sequences

For this section, I define a Cauchy sequence of rational numbers, and then define a real number as a Cauchy sequence of rational numbers. So the real numbers are the collection of all Cauchy sequences of rational numbers.

1. You need to prove the following: If $\{a_n\}$ and $\{b_n\}$ are two Cauchy sequences, then
   (a) $\{-a_n\}$ is a Cauchy sequence.
   (b) $\{a_n + b_n\}$ is a Cauchy sequence.
   (c) $\{a_n b_n\}$ is a Cauchy sequence.
   (d) If $\{a_n\} \neq 0$, then $\{\frac{1}{a_n}\}$ is a Cauchy sequence.
2. You need to prove that this definition of the real numbers satisfies the axioms above.
3. I also want you to prove using the definition of the real numbers from class that every Cauchy sequence of real numbers converges to a limit. To do this, you might want to show that any Cauchy sequence is bounded, and use that to show that it has to converge to a limit.