Exam 1, Topology I

Due September 26th, 2008

Collaboration is allowed. However if you work with anyone, make sure that you indicate it on the exam.

1. Prove that if X is not a Hausdorff topological space, then there is a sequence in X which converges to more than one limit. [Note that it would suffice to find two points x and y in X and show that there is a sequence that converges to both of them.]

2. For any set X, we can define the *finite complement topology* on X in the following way: A subset U of X is open if $X - U$ is either finite or all of X. We saw this in class briefly for the case $X = \mathbb{R}$ the set of real numbers, however, in general this will define a topology on any set X.

   (a) Show that if $X = \{x_1, x_2, \ldots, x_n\}$ is a finite set, then the finite complement topology on X is the same as the discrete topology on X.

   (b) If $X = \mathbb{R}$, is the finite complement topology comparable to the lower limit topology on $\mathbb{R}$? If so, which is finer?

   (c) For $X = \mathbb{R}$, is the finite complement topology Hausdorff? Give a proof one way or the other.

3. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on a set X.

   (a) Does the collection
   $$\mathcal{T}_1 \bigcap \mathcal{T}_2 = \{\, U \,|\, U \in \mathcal{T}_1 \text{ and } U \in \mathcal{T}_2 \,\}$$
   define a topology on X?

   (b) Does the collection
   $$\mathcal{T}_1 \bigcup \mathcal{T}_2 = \{\, U \,|\, U \in \mathcal{T}_1 \text{ or } U \in \mathcal{T}_2 \,\}$$
   define a topology on X?

4. Let X be a topological space and $Y \subset X$. If X is Hausdorff, does that mean Y is Hausdorff in the subspace topology? Give a proof of your answer.

5. Let A be a subset of the topological space X

   (a) If A is open, is it true that $A = Int\left(\overline{A}\right)$?

   (b) If A is closed, is it true that $A = \overline{Int(A)}$?

6. Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on a set X, with $\mathcal{T} \subset \mathcal{T}'$.

   (a) If X is Hausdorff in the topology $\mathcal{T}$, will X be Hausdorff in the topology $\mathcal{T}'$?

   (b) If X is Hausdorff in the topology $\mathcal{T}'$, will X be Hausdorff in the topology $\mathcal{T}$?

7. Let X and Y be topological spaces, and let $f : X \to Y$ be a function between X and Y. What can you say about whether or not f is continuous in the following cases? Can you say if f is open? Can you say if f is closed? Justify your answers.

   (a) X has the discrete topology and Y has any topology.

   (b) X has any topology and Y has the discrete topology.

   (c) X has the indiscrete topology and Y has any topology.

   (d) X has any topology and Y has the indiscrete topology.

8. Show that the product of two Hausdorff spaces is Hausdorff.