Exam 2, Topology I

Due October 31th, 2008

Collaboration is allowed. However if you work with anyone, make sure that you indicate it on the exam.

1. (a) Prove that if $p : X \to Y$ is a continuous surjective map which is either open or closed, then p is a quotient map.
   (b) Prove that the composition of two quotient maps is a quotient map.
   (c) Let X, Y, and Z be topological spaces, and let $f : X \to Y$ and $g : Y \to Z$ be maps. Suppose that f is a quotient map. Prove that g is continuous iff $g \circ f$ is continuous.

2. Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on X. If $\mathcal{T} \subset \mathcal{T}'$ (i.e. $\mathcal{T}'$ is finer than $\mathcal{T}$), what does connectedness of X in one topology imply about connectedness of X in the other.

3. (a) Let $(a, b) \subset \mathbb{R}$ be an open interval with the subspace topology. Prove that (a, b) is homeomorphic to $\mathbb{R}$.
   (b) Let d be that standard Euclidian metric on $\mathbb{R}^n$, and let $\vec{0} \in \mathbb{R}^n$ be the origin. So $d(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\|$. Prove that $B_d(\vec{0}, 1)$ is homeomorphic to $\mathbb{R}^n$.

4. Define an equivalence relation on $\mathbb{R}$ by $x \sim y$ iff $x - y \in \mathbb{Z}$. This gives us a partition $\widehat{\mathbb{R}}$ of $\mathbb{R}$. Describe the identification space $\widehat{\mathbb{R}}$ and describe its topology.

5. Let (X, d_X) and (Y, d_Y) be two metric spaces. Consider the space $X \times Y$.
   (a) Show that the function
   $$d(x_1 \times y_1, x_2 \times y_2) = \sqrt{(d_X(x_1, x_2))^2 + (d_Y(y_1, y_2))^2}$$
   defines a metric on $X \times Y$.
   (b) Show that the function
   $$\rho(x_1 \times y_1, x_2 \times y_2) = d_X(x_1, x_2) + d_Y(y_1, y_2)$$
   defines a metric on $X \times Y$.
   (c) Both metrics induce a topology on $X \times Y$. Do they always induce the same topology? Justify your answer.

6. Let $p : X \to Y$ be a quotient map. Show that if for each $y \in Y$, the set $p^{-1}(\{y\})$ is connected and if Y is connected, then X is connected.