Final Exam, Topology
December 2008

You need to be able to justify your answers to any question.

1. Let $\mathbb{R}$ be the real numbers with the standard topology, and let $\mathbb{R}_\ell$ be the real numnbers with the lower limit topology. Describe all contimuous functions $f : \mathbb{R} \to \mathbb{R}_\ell$.

2. Let x_0 and x_1 be points of the path-connected space X. Show that $\pi_1(X, x_0)$ is abelian if and only if for every pair α and β of paths from x_0 to x_1, we have $\hat{\alpha} = \hat{\beta}$. Recall that an abelian group G with operation $\cdot$ is a group which satisfies $x \cdot y = y \cdot x$ for all $x, y \in G$.

3. Let G be a topological group with operation $\cdot$ and identity element x_0. Let

$$\Omega(G, x_0) = \{\, f \mid f : I \to G,\ f(0) = f(1) = x_0,\ f \text{ continuous } \}$$

denote the set of all loops in G based at x_0. If $f, g \in \Omega(G, x_0)$, let us define a loop $f \otimes g$ by the rule

$$(f \otimes g)(s) = f(s) \cdot g(s).$$

   (i) Show that the operation $\otimes$ makes the set $\Omega(G, x_0)$ into a group.

   (ii) Show that the operation $\otimes$ induces a group operation $\otimes$ on $\pi_1(G, x_0)$.

   (iii) Show that the two group operations $\otimes$ and $*$ on $\pi_1(G, x_0)$ are the same. In other words, show $[f] \otimes [g] = [f] * [g]$. [Hint: Compute $(f * e_{x_0}) \otimes (e_{x_0} * g)$ and remember $f * e_{x_0} \simeq f$.]

   (iv) Show that $\pi_1(G, x_0)$ is an abelian group. In other words, show $[f] * [g] = [g] * [f]$. [Hint: Compute $(f * e_{x_0}) \otimes (e_{x_0} * g)$ and $(e_{x_0} * f) \otimes (g * e_{x_0})$. Use those computations to get the result.]

4. A continuous function $f : X \to Y$ between topological spaces is called a *homotopy equivalence* with *homotopy inverse* $g : Y \to X$ if $f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$. Here $id_X : X \to X$ is the identity function on X, and id_Y is the identity function on Y. For any space Z, let $[X, Z]$ be the set of homotopy classes of continuous functions from X to Z. For a continuous map $h : X \to Z$, let $[h]$ denote the homotopy class of h in $[X, Z]$.

   (i) Given continuous maps $f : X \to Y$ and $g : Y \to Z$, we can define the composition $g \circ f : X \to Z$. Explain why this gives a well-defined operation at the level of homotopy classes,

$$[g] \circ [f] = [g \circ f].$$

   [Hint: You have already proven this fact in a previous homework.]

   (ii) Using part (i), show that for continuous maps $f : X \to Y$, $g : Y \to Z$, and $h : Z \to W$ we have

$$[id_Y] \circ [f] = [f] = [f] \circ [id_X] \quad \text{and} \quad ([h] \circ [g]) \circ [f] = [h] \circ ([g] \circ [f]).$$

For the last two parts, let $f : X \to Y$ be a homotopy equivalence with homotopy inverse $g : Y \to X$.

   (iii) Show that for any topological space Z, there is a bijection between the set $[Z, Y]$ and the set $[Z, X]$. [Hint: Define a function $\phi : [Z, Y] \to [Z, X]$ by $\phi([h]) = [g] \circ [h]$.]

(iv) Show that for any topological space W, there is a bijection between the set $[X, W]$ and the set $[Y, W]$.

5. Let $K = \{\frac{1}{n} | n \in \mathbb{Z}_+\}$, and let $\mathcal{T}_K$ be the topology of $\mathbb{R}$ in which the basic open sets are open intervals (a, b) and sets of the form $(a, b) - K$. Let $\mathbb{R}_K$ denote the real numbers with the topology $\mathcal{T}_K$.

   (i) Is $\mathbb{R}_K$ a Hausdorff space?

   (ii) Is $\mathbb{R}_K$ a compact space?

   (iii) Is $\mathbb{R}_K$ a connected space?

6. Show that if $h : S^1 \to S^1$ is nullhomotopic, then h has a fixed point and h maps some point x to its antipode $-x$.

7. Let $f, g : X \to Y$ be continuous maps between topological spaces, and let Y be Hausdorff. Show that $\{x \in X \,|\, f(x) = g(x)\}$ is a closed subset of X.

8. Let $p : X \to Y$ be a closed continuous surjective map such that $p^{-1}(\{y\})$ is compact for every $y \in Y$. Such a map p is sometimes called a perfect map. Show that if X is Hausdorff, then so is Y.

9. Let $f : X \to X$ be continuous.

   (i) Show that if $X = [0, 1]$, there is a point x such that $f(x) = x$. The point x is called a *fixed point* of f.

   (ii) What happens if $X = [0, 1)$ or $X = (0, 1)$? Does your argument still work?