Homework 1, Topology I

Due September 8th, 2008

1. Let $\mathbb{R}$ be the set of real numbers, and let $\mathbb{Z}$ be the set of integers. For each of the following subsets of $\mathbb{R} \times \mathbb{R}$, determine whether it is equal to the cartesian product of two subsets of $\mathbb{R}$. Justify your answer.
   (a) $\{(x, y) \mid x \in \mathbb{Z}\}$.
   (b) $\{(x, y) \mid y > x\}$.
   (c) $\{(x, y) \mid x \notin \mathbb{Z} \text{ and } y \in \mathbb{Z}\}$.
   (d) $\{(x, y) \mid x^2 + y^2 < 4\}$.

2. Here is a "proof" that every relation which is both symmetric and transitive is also reflexive: "Since $\sim$ is symmetric, $a \sim b$ implies $b \sim a$. Since $\sim$ is transitive, $a \sim b$ and $b \sim a$ together imply $a \sim a$ as desired." Where is the flaw in this argument?

3. Recall that an ordered set A has the **least upper bound property** if every nonempty subset A_0 of A that is bounded above has a least upper bound. An ordered set A has the **greatest lower bound property** if every nonempty subset A_0 of A which is bounded below has a greatest lower bound. Prove the following
   Theorem: If an ordered set A has the least upper bound property, then it has the greatest lower bound property.

4. Assume that the real line, the set $[0, 1]$, and the set $[0, 1)$ all have the least upper bound property. Does $[0, 1] \times [0, 1]$ with the dictionary ordering have the least upper bound property? What about $[0, 1] \times [0, 1)$? How about $[0, 1) \times [0, 1]$?

5. Let $X = \{a, b, c, d, e\}$. Give three separate topologies $\mathcal{T}_1$, $\mathcal{T}_2$, and $\mathcal{T}_3$ on X so that none of them is comparable with the other two.

6. Show that the collection
$$\mathcal{C} = \{ (a, b) \mid a < b,\ a \text{ and } b \text{ rational}\}$$
   is a basis that generates the standard topology on $\mathbb{R}$.

7. Let $\mathcal{B}$ be the collection of all half-open intervals $[a, b)$ in $\mathbb{R}$. $\mathcal{B}$ is a basis for a topology on $\mathbb{R}$. Let $\mathcal{T}_\ell$ be the topology generated by $\mathcal{B}$. This is called the half-open topology on $\mathbb{R}$ (It is also called the lower limit topology on $\mathbb{R}$). Show that the collection
$$\mathcal{C} = \{ [a, b) \mid a < b,\ a \text{ and } b \text{ rational}\}$$
   is a basis that generates a topology different from the lower limit topology on $\mathbb{R}$.