Homework 2, Topology I

Due September 17th, 2008

1. The lower limit topology on the set $\mathbb{R}$, which we will denote $\mathcal{T}_\ell$, is generated by the basis

$$\{\, [a,b) \,|\, a < b,\ a \in \mathbb{R},\ b \in \mathbb{R} \,\}.$$

   There is another topology on $\mathbb{R}$, which we will denote $\mathcal{T}'$, generated by the basis

$$\{\, (a,b] \,|\, a < b,\ a \in \mathbb{R},\ b \in \mathbb{R} \,\}.$$

   Prove that the topologies $\mathcal{T}_\ell$ and $\mathcal{T}'$ are not comparable, but that both are strictly finer than the standard topology.

2. Consider the set of real numbers $\mathbb{R}$, and let $\mathbb{R}_\ell$ denote the real numbers with the lower limit topology. Recall that the lower limit topology is the topology generated by the basis

$$\{\, [a,b) \,|\, a < b,\ a \in \mathbb{R},\ b \in \mathbb{R} \,\}.$$

   Let $\mathbb{R}$ denote the real numbers with the standard topology, which is generated by the basis

$$\{\, (a,b) \,|\, a < b,\ a \in \mathbb{R},\ b \in \mathbb{R} \,\}.$$

   (a) Draw a picture of a basic open set in the product topology on $\mathbb{R} \times \mathbb{R}_\ell$. Note that you can draw it as a subset of the Euclidian plane.

   (b) Draw a picture of a basic open set in the product topology on $\mathbb{R}_\ell \times \mathbb{R}_\ell$.

   (c) If L is a straight line in the plane, describe the topology that L inherits as a subspace of $\mathbb{R} \times \mathbb{R}_\ell$. Is it a topology we've already seen? In other words, since a straight line looks like the real line, does the subspace topology on L look like a topology we've already seen on $\mathbb{R}$?

   (d) If L is a straight line in the plane, describe the topology that L inherits as a subspace of $\mathbb{R}_\ell \times \mathbb{R}_\ell$. Is it a topology we've already seen? In other words, since a straight line looks like the real line, does the subspace topology on L look like a topology we've already seen on $\mathbb{R}$?

   [Hint: For the last two parts of this problem, you might want to consider the following cases separately: 1) L is a vertical line, 2) L is a horizontal line, and 3) L is neither vertical nor horizontal.]

3. Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on the set X with $\mathcal{T} \subsetneq \mathcal{T}'$. What can you say about the corresponding subspace topologies on the subset Y of X?

4. Consider the set $Y = [-1, 1]$ as a subspace of $\mathbb{R}$ with the standard topology. Which of the following sets are open in Y? Which are open in $\mathbb{R}$?

   (a) $A = \{\, x \,|\, \frac{1}{2} < |x| < 1 \,\}$

   (b) $B = \{\, x \,|\, \frac{1}{2} < |x| \leq 1 \,\}$

   (c) $C = \{\, x \,|\, \frac{1}{2} \leq |x| < 1 \,\}$

   (d) $D = \{\, x \,|\, \frac{1}{2} \leq |x| \leq 1 \,\}$

   (e) $E = \{\, x \,|\, 0 < |x| < 1 \text{ and } \frac{1}{x} \notin \mathbb{Z}_+ \,\}$

5. Let A and B be subsets of a topological space X. Prove the following

   (a) $\overline{A \bigcup B} = \overline{A} \bigcup \overline{B}$

   (b) $\overline{A \bigcap B} \subseteq \overline{A} \bigcap \overline{B}$

   (c) $A^\circ \bigcup B^\circ \subseteq (A \bigcup B)^\circ$

   (d) $A^\circ \bigcap B^\circ = (A \bigcap B)^\circ$

   Give counterexamples to equality in (b) and (c).