Homework 3, Topology I

Due October 20th, 2008

1. Let X be a metric space with metric d. Show that $d : X \times X \to \mathbb{R}$ is continuous. Here $\mathbb{R}$ is the real numbers with the standard topology, and $X \times X$ has the product topology.

2. Let A be a subset of a topological space X. Prove

$$\overline{A} \bigcap \overline{(X - A)} = \overline{A} - A^{\circ}.$$

   This shows that the two descriptions we gave in class for the boundary of A are equivalent.

3. Let X and Y be topological spaces. Prove that a bijective function $f : X \to Y$ is open if and only if it is closed. Prove that this is not true for a general function.

4. Show that the Euclidian metric d on $\mathbb{R}^n$ is a metric, as follows: If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $c \in \mathbb{R}$ define

$$\begin{aligned}
\|\mathbf{x}\| &= \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2}. \\
\mathbf{x} + \mathbf{y} &= (x_1 + y_1, \ldots, x_n + y_n). \\
c\mathbf{x} &= (cx_1, \ldots, cx_n). \\
\mathbf{x} \cdot \mathbf{y} &= x_1 y_1 + \cdots + x_n y_n. \\
d(\mathbf{x}, \mathbf{y}) &= \|\mathbf{x} - \mathbf{y}\| \\
&= \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2}.
\end{aligned}$$

   (a) Prove that $\mathbf{x} \cdot (\mathbf{y} + \mathbf{z}) = \mathbf{x} \cdot \mathbf{y} + \mathbf{x} \cdot \mathbf{z}$.

   (b) Prove that $|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\| \|\mathbf{y}\|$. [HINT: If $\mathbf{x}, \mathbf{y} \neq 0$, let $a = 1/\|\mathbf{x}\|$ and $b = 1/\|\mathbf{y}\|$, and use that fact that $\|a\mathbf{x} \pm b\mathbf{y}\| \geq 0$.]

   (c) Show that $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$. [HINT: Compute $(\mathbf{x} + \mathbf{y}) \cdot (\mathbf{x} + \mathbf{y})$ and apply part (b).]

   (d) Verify that d is a metric on $\mathbb{R}^n$.