Quiz 4, **Calc 2**
February 17, 2006

1. Evaluate

$$\int_0^{\pi/16} \sin^2(4x)dx$$

Solution: To solve this problem, we need to use the half angle formula $\sin^2 x = \frac{1}{2}(1 - \cos(2x))$. Since we are dealing with $\sin^2(4x)$, this gives

$$\begin{aligned}
\int_0^{\pi/16} \sin^2(4x)dx &= \int_0^{\pi/16} \frac{1}{2}(1 - \cos(8x))dx \\
&= \frac{1}{2}\int_0^{\pi/16}(1 - \cos(8x))dx \\
&= \frac{1}{2}\left(x - \frac{1}{8}\sin(8x)\right)\Big|_0^{\pi/16} \\
&= \frac{1}{2}\left(\frac{\pi}{16} - \frac{1}{8}\sin\left(\frac{8\pi}{16}\right)\right) - \frac{1}{2}\left(0 - \frac{1}{8}\sin(0)\right) \\
&= \frac{\pi}{32} - \frac{1}{16}.
\end{aligned}$$

2. Evaluate

$$\int \sin^2 x \cos^3 x \; dx$$

Solution: This is an integral of powers of sin and cosine with an odd power of cosine. The strategy here is to pull out one cosine and set up the substitution $u = \sin x$ and $du = \cos x dx$. We will use the trig identity $\cos^2 x = 1 - \sin^2 x$ to get

$$\begin{aligned}
\int \sin^2 x \cos^3 x \ dx &= \int \sin^2 x \cos^2 x \cos x dx \\
&= \int \sin^2 x (1 - \sin^2 x) \cos x dx \\
&= \int u^2 (1 - u^2) du \\
&= \int (u^2 - u^4) du \\
&= \frac{1}{3} u^3 - \frac{1}{5} u^5 + C \\
&= \frac{1}{3} \sin^3 x - \frac{1}{5} \sin^5 x + C.
\end{aligned}$$

3. Evaluate

$$\int \frac{x^3}{\sqrt{4 + x^2}} dx$$

Solution: In this problem, we see the term $\sqrt{4 + x^2}$ is part of the integrand, so we can try a trig substitution. In this case, we want to use the substitution $x = 2 \tan \theta$ and $dx = 2 \sec^2 \theta d\theta$. That way

$$\begin{aligned}
4 + x^2 &= 4 + 4 \tan^2 \theta \\
&= 4(1 + \tan^2 \theta) \\
&= 4(\sec^2 \theta)
\end{aligned}$$

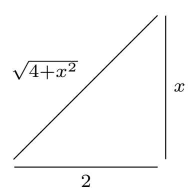
by using a trig identity. After the substitution our integral becomes

$$\begin{aligned}
\int \frac{x^3}{\sqrt{4 + x^2}} dx &= \int \frac{(2 \tan \theta)^3}{\sqrt{4 \sec^2 \theta}} 2 \sec^2 \theta d\theta \\
&= \int \frac{8 \tan^3 \theta}{2 \sec \theta} 2 \sec^2 \theta d\theta \\
&= 8 \int \tan^3 \theta \sec \theta d\theta
\end{aligned}$$

To evaluate the new integral we have in terms of θ, we will pull out one of the copies of $\tan\theta$ to set up the substitution $u = \sec\theta$ and $du = \sec\theta\tan\theta d\theta$.

$$
\begin{aligned}
8\int \tan^3\theta\sec\theta d\theta &= 8\int \tan^2\theta\sec\theta\tan\theta d\theta \\
&= 8\int (\sec^2\theta - 1)\sec\theta\tan\theta d\theta \\
&= 8\int (u^2-1)du \\
&= 8\left(\frac{1}{3}u^3 - u\right) + C \\
&= \frac{8}{3}\sec^3\theta - 8\sec\theta + C
\end{aligned}
$$

Now we need to rewrite our solution in terms of x. To do this, remember our original trig substitution $x = 2\tan\theta$. So $\tan\theta = x/2$. We draw the trangle



This gives us $\sec\theta = \frac{\sqrt{4+x^2}}{2}$. So the final answer is

$$
\int \frac{x^3}{\sqrt{4+x^2}}dx = \frac{8}{3}\left(\frac{\sqrt{4+x^2}}{2}\right)^3 - 8\left(\frac{\sqrt{4+x^2}}{2}\right) + C.
$$