Quiz 8, Calc 2

April 7, 2006

1. Determine of the following series are convergent or divergent and justify your answer.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{2-\sin(n)}{n^3}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}n}{3^n}$

(c) $\displaystyle\sum_{n=11}^{\infty} \frac{5n+3}{12n^2-101n}$

Solution:(a) This series almost looks like $\sum \frac{2}{n^3}$, so we should try a comparison. We know that $-1 \leq \sin(n) \leq 1$ for all n, so it isn't hard to get that $2-\sin(n) \leq 3$. This means $\frac{2-\sin(n)}{n^3} \leq \frac{3}{n^3}$. Now $\sum \frac{3}{n^3}$ converges by p-series, and so our original series converges by the Comparison Test with $\sum \frac{3}{n^3}$.

(b)This series is alternating because of the $(-1)^n$, and so we can try the alternating series test. We just need to check two conditions. First, we need to find

$$\lim_{n\to\infty} \frac{n}{3^n} = 0.$$

This is not hard to check using L'hopital's rule on the function $f(x) = \frac{x}{3^x}$, or by remembering that exponentials grow much faster than polynomials. The second condition we have to check is that the absolute value of the terms is decreasing.

$$\begin{aligned} \frac{(n+1)}{3^{n+1}} &\leq \frac{n}{3^n} \\ 3^n(n+1) &\leq 3^{n+1}n \\ (n+1) &\leq 3n \end{aligned}$$

Since the last line is obviously true, the first line has to be true. So the series is convergent be the alternating series test.

(c) This series has terms that are a rational function in n. So we are going to compare it to a p-series. To figure out what to compare it to, take the highest power of n from the top and put it over the highest power of n from the bottom. This gives

$$\frac{n}{n^2} = \frac{1}{n}.$$

So this series should behave like $\sum \frac{1}{n}$. In order to show this, we can use the Limit Comparison Test.

$$\begin{aligned}
\lim_{n\to\infty} \frac{\frac{5n+3}{12n^2-101n}}{\frac{1}{n}} &= \lim_{n\to\infty} \frac{5n+3}{12n^2-101n}\frac{n}{1} \\
&= \lim_{n\to\infty} \frac{5n^2+3n}{12n^2-101n} \\
&= \frac{5}{12}.
\end{aligned}$$

Since the limit is a finite real number between 0 and ∞, the two series do the same thing. Therefore, our series diverges by Limit compparison test with $\sum \frac{1}{n}$.

2. How many terms do you need to use to approximate the sum

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4+1}$$

to four decimal places (error < 0.0001)?

Solution: This is an alternating series, and for alternating series we have a nice bound for the remainder (error) of a partial sum. The formula is

$$|R_k| \leq b_{k+1}.$$

In our case, this gives

$$|R_k| \leq \frac{1}{(k+1)^4+1}.$$

So if we want to have the error less than 0.0001, we just need to solve the following inequality for k.

$$\begin{aligned}
\frac{1}{(k+1)^4+1} &< \frac{1}{10000} \\
10000 &< (k+1)^4+1 \\
10^4 &< (k+1)^4+1
\end{aligned}$$

It is clear from the last line that the inequality will hold for k equal to 9 or bigger. So we need to use at least 9 terms.