Math 222 **Test 1** **February 19, 2007**

1. (20 points) Find the indicated derivatives. Show your work.

(a) $\dfrac{d}{dx}\sin^{-1}(e^x) =$

Solution: $\dfrac{d}{dx}\sin^{-1}(e^x) = \dfrac{1}{\sqrt{1-(e^x)^2}} \cdot e^x = \dfrac{e^x}{\sqrt{1-e^{2x}}}$

(b) If $f(x) = x\tan^{-1}(x)$ then $f'(1) =$

Solution: $f'(x) = 1 \cdot \tan^{-1}(x) + x \cdot \dfrac{1}{1+x^2} = \tan^{-1}(x) + \dfrac{x}{1+x^2}$,

so $f'(1) = \tan^{-1}(1) + \dfrac{1}{1+1^2} = \dfrac{\pi}{4} + \dfrac{1}{2}$

(c) $D_x\sqrt{\ln x} =$

Solution: $D_x\sqrt{\ln x} = D_x\left[(\ln x)^{\frac{1}{2}}\right] = \dfrac{1}{2} \cdot (\ln x)^{-\frac{1}{2}} \cdot \dfrac{1}{x} = \dfrac{1}{2x\,(\ln x)^{1/2}} = \dfrac{1}{2x\sqrt{\ln x}}$

(d) $\dfrac{dy}{dx}$, if $e^x e^y = \pi$

Solution: There are many ways to do this. If you use implicit differentiation and the product rule you get $\dfrac{d}{dx}\left(e^x e^y\right) = \left(\dfrac{d}{dx}e^x\right)e^y + e^x\left(\dfrac{d}{dx}e^y\right) = e^x e^y + e^x e^y\dfrac{dy}{dx} = \dfrac{d}{dx}\pi = 0.$

Hence $e^x e^y\dfrac{dy}{dx} = -e^x e^y$ so $\dfrac{dy}{dx} = -\dfrac{e^x e^y}{e^x e^y} = -1$

2. (10 points) Find $\dfrac{dy}{dx}$ by logarithmic differentiation. Show your work. Your answer may involve y.

(a) $y = \dfrac{e^x(\tan x)^4}{\sqrt{1+x^2}}$

Solution: $\ln y = \ln(e^x) + \ln\left((\tan x)^4\right) - \ln\left((1+x^2)^{\frac{1}{2}}\right) = x + 4\ln(\tan x) - \frac{1}{2}\ln(1+x^2)$

so $\dfrac{d}{dx}\ln y = \dfrac{1}{y}\dfrac{dy}{dx} = 1 + 4 \cdot \dfrac{1}{\tan x}\sec^2 x - \dfrac{1}{2} \cdot \dfrac{2x}{1+x^2}$.

Hence $\dfrac{dy}{dx} = y\left[1 + 4\cot x\sec^2 x - \dfrac{x}{1+x^2}\right] = y\left[1 + 4\csc x\sec x - \dfrac{x}{1+x^2}\right]$

(b) $y = (\sin x)^x$

Solution: $\ln y = \ln\left((\sin x)^x\right) = x\ln(\sin x)$ so $\dfrac{d}{dx}\ln y = \dfrac{1}{y}\dfrac{dy}{dx} = 1 \cdot \ln(\sin x) + x \cdot \dfrac{1}{\sin x} \cdot \cos x$.

Hence $\dfrac{dy}{dx} = y\left[\ln(\sin x) + x\cot x\right]$

3. (24 points) Integrate by first making a substitution. You will lose credit if you use a "quick formula" from the book instead of performing the substitution. Show all your work, and indicate clearly your substitutions.

(a) $\displaystyle\int x\sec(x^2)\,dx =$

Solution: Make the substitution $u = x^2$, $du = 2x\,dx$, so $x\,dx = \frac{1}{2}du$:

$$\int x\sec(x^2)\,dx = \frac{1}{2}\int \sec u\,du = \frac{1}{2}\ln|\sec u + \tan u| + C = \frac{1}{2}\ln\left|\sec(x^2) + \tan(x^2)\right| + C$$

(b) $\displaystyle\int_0^{1/4} \frac{dt}{1+16t^2} =$

Solution: Make the substitution $u = 4t$, $du = 4dt$, so $dt = \frac{1}{4}du$ and $1 + 16t^2 = 1 + (4t)^2 = 1 + u^2$. Also, the limits $t = 0$ and $t = 1/4$ become $u = 0$ and $u = 1$:

$$\int_0^{1/4} \frac{dt}{1+16t^2} = \int_0^1 \frac{\frac{1}{4}du}{1+u^2} = \frac{1}{4}\tan^{-1} u\Big|_0^1 = \frac{1}{4}\left[\tan^{-1} 1 - \tan^{-1} 0\right] = \frac{1}{4}\left[\frac{\pi}{4} - 0\right] = \frac{\pi}{16}$$

(c) $\displaystyle\int e^{2x}\cos\left(e^{2x}\right)\,dx =$

Solution: Make the substitution $u = e^{2x}$, $du = 2e^{2x}\,dx$, so $e^{2x}\,dx = \frac{1}{2}du$:

$$\int e^{2x}\cos\left(e^{2x}\right)\,dx = \frac{1}{2}\int \cos u\,du = \frac{1}{2}\sin u + C = \frac{1}{2}\sin\left(e^{2x}\right) + C$$

4. (10 points)

(a) Define: $\ln 2007 =$

Solution: $\displaystyle \ln 2007 = \int_1^{2007} \frac{dt}{t}$

(b) Solve for x: $e^{2x} + e^x = 2$

Solution: Rewrite this as $e^{2x} + e^x - 2 = 0$, so $(e^x)^2 + e^x - 2 = 0$. This is a quadratic equation in e^x, which factors as $(e^x + 2)(e^x - 1) = 0$. Hence either $e^x = -2$ or $e^x = 1$. The first alternative can't happen, since e^x is never negative. The second alternative gives $x = \ln 1 = 0$.

(c) Solve for x: $\ln x + \ln 7 = \ln(x + 7)$

Solution: Since $\ln x + \ln 7 = \ln(7x)$ this equation becomes $\ln(7x) = \ln(7 + x)$, so $e^{\ln(7x)} = e^{\ln(7+x)}$, which is the same as $7x = 7 + x$. Solve this for x to get $x = 7/6$

5. (21 points) Find the limits. Show all your work. Identify any use of L'Hôpital's Rule, and indicate briefly why it is applicable.

(a) $\displaystyle\lim_{x\to 0} \frac{\cos 2x - \cos 3x}{x^2} =$

Solution: Use L'Hôpital's Rule twice (indicated by "=*"); in both cases the indeterminate form is $\frac{0}{0}$:

$$\lim_{x\to 0} \frac{\cos 2x - \cos 3x}{x^2} =^* \lim_{x\to 0} \frac{-2\sin 2x + 3\sin 3x}{2x} =^* \lim_{x\to 0} \frac{-4\cos 2x + 9\cos 3x}{2} = \frac{-4+9}{2} = \frac{5}{2}$$

(b) $\displaystyle\lim_{x\to +\infty} (1 + 2x)^{3/x} =$

Solution: Let $y = (1 + 2x)^{3/x}$ so $\ln y = \dfrac{3}{x} \cdot \ln(1 + 2x)$. Then use L'Hôpital's Rule (indicated by "=*"); the indeterminate form is $\frac{\infty}{\infty}$:

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{3\ln(1+2x)}{x} =^* \lim_{x\to\infty} \frac{3 \cdot \frac{1}{1+2x} \cdot 2}{1} = \lim_{x\to\infty} \frac{6}{1+2x} = 0.$$

Hence, since the exponential function is continuous,
$\lim_{x\to+\infty}(1+2x)^{3/x} = \lim_{x\to+\infty} y = \lim_{x\to+\infty} e^{\ln y} = e^{\lim_{x\to+\infty}\ln y} = e^0 = 1$

(c) $\lim_{n\to+\infty}(\ln(2n-1)-\ln n) =$

Solution: Use continuity of the logarithm function: $\lim_{n\to+\infty}(\ln(2n-1)-\ln n) = \lim_{n\to+\infty}\ln\left(\frac{2n-1}{n}\right) = \ln\left(\lim_{n\to+\infty}\frac{2n-1}{n}\right) = \ln\left(\lim_{n\to+\infty}\left(2-\frac{1}{n}\right)\right) = \ln 2$

6. (10 points) Mr. Lin invested some money in 1970 in a bond having a fixed interest rate, so his investment M satisfies the differential equation $\frac{dM}{dt} = kM$ where k is the interest rate (assuming continuous compounding.) In 1980 his investment was worth \$3000 and in 2000 it was worth \$27000.

(a) How much will his investment be worth in 2020?
Solution: His investment grew by a factor of 9 in the twenty years from 1980 to 2000, so it will grow by another factor of 9 in the twenty years from 2000 to 2020. So in 2020 his investment will be worth $9 \times \$27000 = \$243000$

(b) What was his original investment in 1970?
Solution: His investment grows by a factor of 9 every twenty years. In a period of ten years (which is half of twenty) it will grow by a factor of $\sqrt{9} = 3$. Hence his investment in 1970 was one third of its value in 1980, or $\$3000/3 = \$1000$

(c) Find the interest rate on his investment.
Solution: Measuring time in years from 1970 we have $M(t) = M_0 e^{kt}$, where M_0, the initial investment, is \$1000. Also, in 1980, when $t = 10$, we have $M(10) = \$3000 = \$1000 \times e^{10k}$. Divide this by \$1000 to get $e^{10k} = 3$ and then take logarithms to get $10k = \ln 3$. Hence $k = \frac{1}{10}\ln 3$. This is about 0.1098612289, or approximately 11%

7. (5 points) This problem concerns the **sequence** $\left\langle \frac{(n-4)^2}{n^2} \right\rangle$ for $n \geq 1$. Circle your answers, and fill in the box if you can. You do not have to show any work.

(a) Does this sequence have a limit? | yes | no | If yes, what is it?

Solution: Yes. The limit is 1.

(b) Does this sequence converge? | yes | no |

Solution: Yes

(c) This sequence is | increasing | decreasing | neither increasing nor decreasing |.

Solution: Neither

(d) Is this sequence bounded? | yes | no |

Solution: Yes