Contents

1. Introduction

1. $\dfrac{d}{dx}\left(\cos(x^2)\right) = -2x\sin(x^2)$

2. $\dfrac{d}{dx}\left(\sqrt{\sin(x)}\right) = \dfrac{1}{2}\left(\sin(x)\right)^{-1/2}\cos(x)$

3. $\dfrac{d}{dx}\left(1-\sqrt{x}\right) = -\dfrac{1}{2}x^{-1/2}$

4. $\dfrac{d}{dx}\left(x^3\ln(x)\right) = 3x^2\ln(x) + x^2$

5. $\dfrac{d}{dx}\left(\sqrt{1+\tan(x)}\right) = \dfrac{1}{2}\left(1+\tan(x)\right)^{-1/2}\sec^2(x)$

6. $\dfrac{d}{dx}\left(\dfrac{1}{\cos(x^2)}\right) = \dfrac{2x\sin(x^2)}{\cos^2(x^2)}$

7. $\dfrac{d}{dx}\left(\cos\left(\sin(x)\right)\right) = -\sin\left(\sin(x)\right)\cos(x)$

8. $\dfrac{d}{dx}\left(\tan\left(\ln(x)\right)\right) = \sec^2\left(\ln(x)\right)\dfrac{1}{x}$

9. $\dfrac{d}{dx}\left(\arctan\left(\sqrt{x}\right)\right) = \dfrac{1}{1+x}\cdot\dfrac{1}{2}x^{-1/2}$

10. $y = -\cos(1+x) + C$

11. $y = \dfrac{2}{5}x^{5/2} + C$

12. $y = \dfrac{1}{2}\sin\left(x^2\right) + C$

13. $y = \ln|\sec(x)| + C$

14. $y = \arctan(x+1) + C$

15. $y = \ln|\csc(x) - \cot(x)| + C$

16. $y = x\ln(x) - x + C$

17. $y = xe^x - e^x + C$

18. $y = \ln|\tan(x)| + C$

19. $y = 1 + e^x$

20. $y = 2 - \cos(x)$

21. $y = 3 + \ln(x)$

22. $y = 1 + \sin(x)$

2. Separable Equations

1. $4x^2 + y^2 = C$
2. $y^2 = \ln(x^2 + 1) + C$
3. $y = \dfrac{3}{x^3 + C}$
4. $y = Ce^{x^2/2}$
5. $y = Ce^{x + \frac{1}{3}x^3}$
6. $y^2 + y = \dfrac{1}{2}x^2 + C$
7. $\sin(x) + \cos(y) = C$
8. $x^2 - y^2 = C$
9. $y = \tan(x + C)$
10. $y = -\dfrac{1}{2} + \dfrac{7}{2}e^{2x}$
11. $y = \dfrac{2}{2 - 4x - x^2}$
12. $x^2 + y^2 = 4$
13. $y = \dfrac{1}{2}\tan(2x)$
14. $y = \tan(x^2 + 2x)$
15. $y^2 + 3y = \sin(2x) - 2$
16. $y = \dfrac{1}{x + 2}$
17. $\frac{1}{2}y^2 + y = \ln(x + 1) + \frac{3}{2} - \ln(2)$
18. $y = 2e^{e^x}$
19. $y^2 = 1 + \sqrt{x^2 - 16}$

3. Linear Algebra

1.

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

2.

$$\begin{pmatrix} 3 & 3 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 2 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 5 & 10 & 15 \\ 0 & 1 & 2 \end{pmatrix}$$

3.

$$A + D = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 4 & 2 \end{pmatrix}$$

$$AB = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 4 & 7 \\ 1 & 2 & 3 \end{pmatrix}$$

$$AD = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 8 & 5 \\ 2 & 1 \end{pmatrix}$$

$$BC = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 9 & 5 \\ 5 & 2 \end{pmatrix}$$

$$CA = \begin{pmatrix} 1 & 2 \\ 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 1 & 2 \\ 3 & 4 \end{pmatrix}$$

$$CB = \begin{pmatrix} 1 & 2 \\ 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 4 & 7 \\ 1 & 2 & 3 \\ 2 & 5 & 8 \end{pmatrix}$$

$$CD = \begin{pmatrix} 1 & 2 \\ 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 8 & 5 \\ 2 & 1 \\ 7 & 4 \end{pmatrix}$$

$$DA = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix}$$

$$DB = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 5 & 8 \\ 3 & 8 & 13 \end{pmatrix}$$

4. Determinants – Eigenvectors

1.

$$\det\left(\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\right) = -2$$

2.

$$\det\left(\begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & 0 \\ 1 & 1 & 1 \end{pmatrix}\right) = -5$$

3. This matrix has eigenvalues $\lambda = 0$, $\lambda = -2$, and $\lambda = 3$.

   Any eigenvector of this matrix with eigenvalue $\lambda = 0$ is of the form

$$c_1 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

   for some nonzero scalar c_1.

   Any eigenvector of this matrix with eigenvalue $\lambda = -2$ is of the form

$$c_2 \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

   for some nonzero scalar c_3.

   Any eigenvector of this matrix with eigenvalue $\lambda = 3$ is of the form

$$c_3 \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

   for some nonzero scalar c_3.

4. The eigenvalues are $\lambda = 1$ and $\lambda = 4$.

   Any eigenvector of this matrix with eigenvalue $\lambda = 1$ is of the form

$$c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

   for some nonzero scalar c_1.

Any eigenvector of this matrix with eigenvalue $\lambda = 4$ is of the form

$$c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

for some nonzero scalar c_2.

5. The eigenvalues are $\lambda = 1$ and $\lambda = 6$.

   Any eigenvector of this matrix with eigenvalue $\lambda = 1$ is of the form

$$c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

   for some nonzero scalar c_1.

   Any eigenvector of this matrix with eigenvalue $\lambda = 6$ is of the form

$$c_2 \begin{pmatrix} 2 \\ 7 \end{pmatrix}$$

   for some nonzero scalar c_2.

6. The eigenvalues are $\lambda = 2$ and $\lambda = 4$.

   Any eigenvector of this matrix with eigenvalue $\lambda = 2$ is of the form

$$c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

   for some nonzero scalar c_1.

   Any eigenvector of this matrix with eigenvalue $\lambda = 4$ is of the form

$$c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

   for some nonzero scalar c_2.

7. This matrix has no (real) eigenvalues.

8. This matrix has eigenvalues $\lambda = 1$, $\lambda = 2$, and $\lambda = 3$.

   Any eigenvector of this matrix with eigenvalue $\lambda = 1$ is of the form

$$c_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

   for some nonzero scalar c_1.

   Any eigenvector of this matrix with eigenvalue $\lambda = 2$ is of the form

$$c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

   for some nonzero scalar c_3.

Any eigenvector of this matrix with eigenvalue $\lambda = 3$ is of the form

$$c_3 \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

for some nonzero scalar c_3.

9. This matrix has eigenvalues $\lambda = -1/2$ and $\lambda = -3/2$.

Any eigenvector of this matrix with eigenvalue $\lambda = -1/2$ is of the form

$$c_1 \begin{pmatrix} 3 \\ 10 \end{pmatrix}$$

for some nonzero scalar c_1.

Any eigenvector of this matrix with eigenvalue $\lambda = -3/2$ is of the form

$$c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

for some nonzero scalar c_2.

5. The Chain Rule – Picard's Theorem

1. $\dfrac{du}{dt} = \dfrac{1}{2}(e^{3t} + e^t)^{-1/2}(3e^{3t} + e^t)$

2. $\dfrac{du}{dt} = -\csc^2(t)$

3. $\dfrac{du}{dt} = u\dfrac{4\ln(t)}{t}$

4. $\dfrac{du}{dt} = -\dfrac{1}{2}\dfrac{2t + 4t^3}{(t^2 + t^4)^{3/2}}$

5. $\dfrac{du}{dt} = 2t\cos(t) - t^2\sin(t) + \cos(t) - t\sin(t) + 3t^2$

6. $y_2 = 1 - x + x^2 - \frac{1}{6}x^3$

7. $y_2 = \frac{1}{2}x^2 - \frac{1}{6}x^3$

8. $y_2 = 1 + \frac{1}{2}x^2 + \frac{1}{4}x^4 + \frac{1}{24}x^6$

9. $y_2 = \dfrac{1}{3} + \dfrac{2}{3}\left(\dfrac{1}{3} + \dfrac{2}{3}x^{3/2}\right)^{3/2}$

10. $y_2 = 0$

6. Exact Differential Equations

1. This equation is exact and the solution is $xy + \frac{1}{2}y^2 = C$.
2. This equation is exact and the solution is $y\ln(x) = C$.
3. This equation is exact and the solution is $\dfrac{x^2+y^2}{y} = C$.
4. This equation is not exact.
   - The integrating factor $F(x) = e^x$ gives the exact differential equation
     $$e^x \sin(y)\, dx + e^x \cos(y)\, dy = 0$$
     with solution
     $$e^x \sin(y) = C\ .$$
   - The integrating factor $G(y) = \csc(y)$ gives the exact differential equation
     $$dx + \cot(y)\, dy = 0$$
     with solution
     $$x + \ln|\sin(y)| = C$$
5. This equation is not exact.
   - The integrating factor $F(x) = e^{2x}$ gives the exact differential equation
     $$2e^{2x}\cos(y)\, dx - e^{2x}\sin(y)\, dy = 0$$
     with solution
     $$e^{2x}\cos(y) = C\,.$$
   - The integrating factor $G(y) = \sec(y)$ gives the exact differential equation
     $$2\, dx - \tan(y)\, dy = 0$$
     with solution
     $$2x - \ln|\sec(y)| = C\,.$$
6. This equation is not exact.
   - The integrating factor $F(x) = \dfrac{1}{x^2}$ gives the exact differential equation
     $$\frac{1}{x}\, dy - \frac{y-x}{x^2}\, dx = 0$$
     with solution
     $$\frac{y}{x} + \ln|x| = C$$

7. This equation is not exact.

   - The integrating factor $G(y) = y^2$ gives the exact differential equation
   $$2xy^3\,dx + (3x^2y^2 + 2y^3)\,dy = 0$$
   with solution
   $$x^2y^3 + \frac{1}{2}y^4 = C\,.$$

8. This equation is not exact.

   - The integrating factor $F(x) = e^{x^2}$ gives the exact differential equation
   $$2xye^{x^2}\,dx + e^{x^2}\,dy = 0$$
   with solution
   $$ye^{x^2} = C\,.$$
   - The integrating factor $G(y) = \dfrac{1}{y}$ gives the exact differential equation
   $$2x\,dx + \frac{1}{y}\,dy = 0$$
   with solution
   $$x^2 + \ln|y| = C$$

9. This equation is exact and the solution is $yx^{-1} = C$.

10. 2, 4, 5, 8, and 9 can all be solved by separation of variables.

7. Substitutions – Orthogonal Families

1.
$$\frac{5-\sqrt{5}}{10}\ln\left(\frac{y}{x}+\frac{1+\sqrt{5}}{2}\right)+\frac{5+\sqrt{5}}{10}\ln\left(\frac{y}{x}+\frac{1-\sqrt{5}}{2}\right)=-\ln(x)+\frac{\sqrt{5}}{10}\ln\left(\frac{3-\sqrt{5}}{3+\sqrt{5}}\right)$$

2. $y = ce^{x^2/2y^2}$

3. $\sec(y/x) + \tan(y/x) = cx$

4. $\dfrac{x}{x-y} = x + c$ or, solving for y, $y = \dfrac{x^2+cx-x}{x+c}$

5. $2x + y^2 = c$

6. $\frac{1}{2}y^2 = c + \ln|\sin(x)|$

7. $\dfrac{1}{2}\ln\left(\dfrac{y^2+xy+x^2}{x^2}\right) + \dfrac{1}{\sqrt{3}}\tan^{-1}\left(\dfrac{2y+x}{x\sqrt{3}}\right) = -\ln|x| + c$

8. $y = ce^{-2x}$

9. $x^2 - y^2 = c$

10. $x^2 + 3y^2 = c$

8. First Order Linear Differential Equations

1. $y = 1 + \frac{C}{x^2}$
2. $y = \frac{4}{3} + Ce^{-3x}$
3. $y = -\frac{1}{5}\cos(x) + \frac{2}{5}\sin(x) + Ce^{-2x}$
4. $y = e^{-x} + Ce^{-2x}$
5. $y = e^{-x}\left(\ln|\sec(x)| + C\right)$
6. $y = C\sec(x)$
7. $y = x(e^x + C)$
8. $y = (x + C)e^{\cos(x)}$
9. $y = \frac{1}{5}x^4 + \frac{C}{x}$
10. $y^2 = -\frac{1}{2}x^2 + Cx^6$
11. $y^3 = \frac{3}{2}e^{-x} + Ce^{-3x}$
12. $y^{-2} = -1 + Ce^{-2x}$
13. $y^{-3} = \frac{3}{4}x^{-1} + Cx^3$

9. Complex Numbers

1. $i^{103578} = -1$
2. $(1+i)^{200} = 2^{100} = 1267650600228229401496703205376$
3. $(1+i)^2 + (1-i)^2 = 0$
4. $z = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$
5. $z = -1 \pm 2i$

10. Second Order Linear Equations

1. The functions 1, x, and x^2 are linearly independent since the Wronskian is $W(x) = 2 \neq 0$.

2. The functions e^x and xe^x are linearly independent since $xe^x/e^x = x$ is not constant.

3. The functions $\cos^2(x)$, $\sin^2(x)$, and $\cos(2x)$ are linearly dependent since $\cos^2(x) - \sin^2(x) - \cos(2x) = 0$.

4. The functions $\cos^2(x)$, $\sin^2(x)$, and $\sin(2x)$ are linearly independent since the Wronskian is $W(x) = -4 \neq 0$.

5. $y = c_1 x + c_2 x \ln|x|$

6. $y = c_1 e^{-x} + c_2 x e^{-x}$

7. $y = c_1 \dfrac{\sin(x)}{x} + c_2 \dfrac{\cos(x)}{x}$

8. $y = c_1 + c_2 \ln|x|$

11. Constant Coefficients

1. $y = c_1 e^x + c_2 e^{2x}$
2. $y = c_1 e^x \cos(x) + c_2 e^x \sin(x)$
3. $y = c_1 e^x \cos(2x) + c_2 e^x \sin(2x)$
4. $y = c_1 \cos(x) + c_2 \sin(x)$
5. $y = c_1 e^x + c_2 x e^x$
6. $y = c_1 e^{2x} + c_2 x e^{2x}$
7. $y = e^{2x} \cos(x) - 4e^{2x} \sin(x)$
8. $y = xe^{-x}$
9. $y = e^x + e^{2x}$
10. $y = e^x + 2e^{-x}$

12. Differential Operators

1. $(D^2 - 3D + 2)y = e^{3x}$
2. $(D^2 - 1)y = \cos(x)$
3. $(D^2 - 2D + 2)y = e^x$
4. $(D^2 + 1)y = e^x$
5. $(D^2 + 1)y = \sin(x)$
6. $(D^2 + 1)y = 1$
7. $(D^2 + 2D + 2)y = e^x$
8. $D^2y = x^2$

13. Particular Solutions

1. $y(x) = \frac{1}{2}e^{3x} + c_1 e^x + c_2 e^{2x}$
2. $y(x) = -\frac{1}{2}\cos(x) + c_1 e^x + c_2 e^{-x}$
3. $y = e^x + c_1 e^x \cos(x) + c_2 e^x \sin(x)$
4. $y = \frac{1}{2}e^x + c_1 \cos(x) + c_2 \sin(x)$
5. $y = -\frac{1}{2}x\cos(x) + c_1 \cos(x) + c_2 \sin(x)$
6. $y = 1 + c_1 \cos(x) + c_2 \sin(x)$
7. $y = \frac{1}{2}x - \frac{1}{2} + c_1 e^{-x}\cos(x) + c_2 e^{-x}\sin(x)$
8. $y = \frac{1}{12}x^4 + c_1 + c_2 x$
9. $y = \frac{1}{3}xe^{2x} + \frac{5}{9}e^{2x} + \frac{4}{9}e^{-x}$
10. $y = \frac{2}{5}\cos(x) + \frac{1}{5}\sin(x) - \frac{2}{5}e^x\cos(x) + \frac{6}{5}e^x\sin(x)$
11. $y = \frac{1}{6}x^3 e^{2x} + c_1 e^{2x} + c_2 x e^{2x}$
12. $y = -\frac{1}{2}x\cos(2x) + \frac{1}{4}\ln|\sin(2x)|\sin(2x) + c_1\cos(2x) + c_2\sin(2x)$
13. $y = -e^{-x}\cos(x) + c_1 e^{-x} + c_2 x e^{-x}$

16. Euler Cauchy Equations

1. $y = c_1 x + c_2 x^2$
2. $y = c_1 \cos(\ln(x)) + c_2 \sin(\ln(x))$
3. $y = c_1 x + c_2 x \ln(x)$
4. $y = c_1 x^2 + c_2 x^5$
5. $y = c_1 x^3 + c_2 x^{-1}$
6. $y = c_1 x^6 + c_2 x^{-2}$
7. $y = \frac{6}{7} x^2 + \frac{15}{7} x^{-1}$
8. $y = x^3$
9. $y = 2 + x^2$
10. $y = x + x^{-1}$
11. $y = -\frac{1}{2} x + c_1 x^2 + c_2 x^{-1}$
12. $y = \frac{1}{2} x \cos(x) - \frac{1}{2} x^2 \sin(x) + \frac{1}{2} x^3 \operatorname{Ci}(x) + c_1 x + c_2 x^3$

17. Systems of Equations

1. $y_1 = -\frac{2}{3}e^t + \frac{2}{3}e^{4t}$, $y_2 = \frac{4}{3}e^t + \frac{2}{3}e^{4t}$
2. $y_1 = e^{4t} + e^{6t}$, $y_2 = -e^{4t} + e^{6t}$
3. $y_1 = 2c_1 + c_2e^{3t}$, $y_2 = -c_1 + c_2e^{3t}$
4. $y_1 = 4c_1e^{3t} - 2c_2e^{-3t}$, $y_2 = c_1e^{3t} + c_2e^{-3t}$
5. $y_1 = c_1e^{2x} + c_2xe^{2x}$, $y_2 = c_1e^{2x} + c_2xe^{2x} + \frac{1}{3}c_2e^{2x}$

18. Power Series

1.
$$y = a_0\left(1 + \frac{1}{4}x^4 - \frac{1}{10}x^6 + \cdots\right) + a_1\left(x - \frac{1}{2}x^3 + \frac{3}{8}x^5 + \cdots\right)$$

$$a_n = -\frac{3}{n(n-1)}\left((n-2)a_{n-2} - a_{n-4}\right)$$

2.
$$y = a_0\left(1 + \frac{1}{2}x^3 - \frac{1}{20}x^6 + \cdots\right) + a_1\left(x + \frac{1}{12}x^4 - \frac{5}{504}x^7 + \cdots\right)$$

$$a_n = -\frac{2n-9}{n(n-1)}a_{n-3}$$

3.
$$y = a_0\left(1 - \frac{1}{4}x^2 - \frac{1}{12}x^3 + \frac{1}{120}x^5 + \frac{1}{288}x^6 + \frac{1}{1120}x^7 + \cdots\right)$$

$$a_n = \frac{1}{2n}\left((n-1)a_{n-1} - a_{n-2}\right)$$

4.
$$y = a_0\left(1 + \frac{1}{4}x^4 - \frac{1}{15}x^6 + \cdots\right) + a_1\left(x - \frac{1}{4}x^3 - \frac{1}{12}x^5 + \cdots\right)$$

$$a_n = -\frac{1}{n(n-1)}\left(2(n-2)a_{n-2} - 3a_{n-4}\right)$$

5.
$$y = a_0\left(1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \cdots\right) + a_1\left(x + \frac{1}{3}x^3 + \frac{1}{60}x^5 + \cdots\right)$$

$$a_n = \frac{1}{n(n-1)}\left((n-1)a_{n-2} - a_{n-4}\right)$$

6.
$$y = a_0\left(1 + \frac{1}{2}x^2 + \frac{7}{24}x^4 + \cdots\right) + a_1\left(x + \frac{5}{6}x^3 + \frac{59}{120}x^5 + \cdots\right)$$

$$a_n = \frac{1}{n(n-1)}\left((4n-7)a_{n-2} - a_{n-4}\right)$$

7.
$$y = a_1 x + a_0 \left(1 + \frac{1}{6}x^3 - \frac{1}{90}x^6 + \frac{1}{1296}x^8 + \cdots\right)$$
$$a_n = -\frac{n-4}{n(n-1)}a_{n-3}$$

8.
$$y = a_1 x + a_2 x^2$$
$$(n^2 - 3n + 2)a_n = 0$$

9.
$$y = a_0 \left(1 + \frac{1}{12}x^4 + \frac{1}{90}x^6 + \cdots\right) + a_1 \left(x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \cdots\right)$$
$$a_n = \frac{n-2}{n(n-1)}a_{n-2} + \frac{1}{n-1}a_{n-4}$$

10.
$$y = a_0 \left(1 + \frac{1}{8}x^2 + \frac{3}{128}x^4 + \cdots\right) + a_1 \left(1 + \frac{1}{6}x^3 + \frac{1}{30}x^5 + \cdots\right)$$
$$a_n = \frac{n-1}{4n}a_{n-2}$$

11.
$$y = a_0(1 + 6x^2 + x^4) + a_1(x + x^3)$$

12.
$$y = a_0(1 - 4x^2 + 2x^4) + a_1 \left(1 - \frac{5}{4}x^3 + \frac{7}{32}x^5 + \frac{3}{128}x^7 + \cdots\right)$$
$$a_n = \frac{(n-6)(n+2)}{2n(n-1)}a_{n-2}$$

13.
$$y = a_0 \left(1 - \frac{1}{12}x^4 + \frac{1}{672}x^8 - \cdots\right) + a_1 \left(x - \frac{1}{20}x^5 + \frac{1}{1440}x^9 - \cdots\right)$$
$$a_n = -\frac{1}{n(n-1)}a_{n-4}$$

19. Laplace Transform

1. $\mathscr{L}(e^{2t}t^3) = \dfrac{6}{(s-2)^4}$

2. $\mathscr{L}(e^t \cos(t)) = \dfrac{s-1}{(s-1)^2+1}$

3. $\mathscr{L}(\cos^2(t)) = \dfrac{s^2+2}{s(s^2+4)}$

4. $\mathscr{L}(\sin^2(t)) = \dfrac{2}{s(s^2+4)}$

5. $\mathscr{L}(\cos(2t)) = \dfrac{s}{s^2+4}$

6. $\mathscr{L}(e^t \sin(2t)) = \dfrac{2}{(s-1)^2+4}$

7. $\mathscr{L}\left(t \sin(t)\right) = \dfrac{2s}{(s^2+1)^2}$

8. $\mathscr{L}(t \cos(t)) = \dfrac{s^2-1}{(s^2+1)^2}$

9. $\mathscr{L}(t^2 \sin(t)) = \dfrac{2(3s^2-1)}{(s^2+1)^3}$

20. Shifting Theorems

1. $\mathscr{L}^{-1}\left(\dfrac{s-1}{s^2+2s+2}\right) = e^{-t}\cos(t) - 2e^{-t}\sin(t)$

2. $\mathscr{L}^{-1}\left(\dfrac{s}{s^2+2s+2}\right) = e^{-t}\cos(t) - e^{-t}\sin(t)$

3. $\mathscr{L}^{-1}\left(e^{-s}\dfrac{1}{1+s^2} + e^{-2s}\dfrac{s}{1+s^2}\right) = u(t-1)\sin(t-1) + u(t-2)\cos(t-2)$

4. $\mathscr{L}^{-1}\left(\dfrac{s}{s^2+10s+29}\right) = e^{-5t}\cos(2t) - \dfrac{5}{2}e^{-5t}\sin(2t)$

5. $\mathscr{L}^{-1}\left(\dfrac{1}{s^2+2s+2}\right) = e^{-t}\sin(t)$

6. $\mathscr{L}\left(f(t)\right) = e^{-s}\dfrac{s+1}{s^2} - 2e^{-2s}\dfrac{2s+1}{s^2} + e^{-3s}\dfrac{3s+1}{s^2}$

7. $\mathscr{L}\left(f(t)\right) = \dfrac{1}{s^2}(1 - e^{-s} - e^{-2s} + e^{-3s})$

8. $\mathscr{L}\left(f(t)\right) = \dfrac{1}{s^2+1}(1 + e^{-\pi s})$

21. Calculating With Laplace

1. $y = -\frac{5}{34}\cos(t) - \frac{3}{34}\sin(t) + \frac{1}{5}te^{-t} + \frac{173}{425}e^{4t} + \frac{37}{50}e^{-t}$

2. $y = -\frac{3}{50}\cos(t) + \frac{2}{25}\sin(t) - \frac{1}{16}e^{t} + \frac{449}{400}e^{-3t} + \frac{87}{20}te^{-3t}$

3. $y = \frac{13}{250} - \frac{3}{25}t + \frac{1}{10}t^2 + \frac{237}{250}e^{-3t}\cos(t) + \frac{1241}{250}e^{-3t}\sin(t)$

22. Problems

Problems on Page 108

1. This set of vectors is linearly independent.
2. This set of vectors is linearly dependent.
3. $\det(A) = 6$
4. $\det(B) = 0$
5. No, the first matrix times the second matrix is
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$
which is not the identity.

Problems on Page 109

1. a. $-\lambda^3 + 3\lambda - 2$

   b. The eigenvalues of A are -2 and 1.

   c. $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector of A with eigenvalue -2; the two vectors $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ are linearly independent eigenvectors of A with eigenvalue 1

2. a. $P_3(x) = 1 + x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6$

   b. $P_3(x) = 2 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$

3. a. $x^2y^2 - 3x + 4y = C$

   b. $\frac{1}{4}x^4 + xy^3 = C$

Exam 1 Review

1. a. i. $\det(A - \lambda I) = \lambda^2 - \lambda - 6$

   ii. Eigenvalues are $\lambda = 3$ and $\lambda = -2$. $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector of A with eigenvalues $\lambda = 3$ and $\begin{pmatrix} -2/3 \\ 1 \end{pmatrix}$ is an eigenvector of A with eigenvalues $\lambda = -2$ and

   b. $\det(B - \lambda I) = -\lambda^3 + 12\lambda + 16$

2. $y_1(x) = e^x - x$
   $y_2(x) = 1 + \frac{1}{2}x^2$
   $y_3(x) = e^x - x - \frac{1}{6}x^3$

3. a. $y = Ce^{-x^2}$

   b. $\ln|y| + y^2 = -\cos(x) + C$

   c. $\ln(e^y + 1) = -\ln(1 + \cos(x)) + 2\ln(2)$, or $y = \ln\left(\frac{3 - \cos(x)}{1 + \cos(x)}\right)$

   d. $y = \dfrac{2x}{1 + Cx^2}$

   e. $y = \dfrac{2x^{3/2}}{1 - 2\sqrt{x}}$

   f. $x^2y^2 - 3x + 4y = C$

   g. $y = \dfrac{2xe^x - 2e^x + 2x^3 + C}{x}$

   h. $x^4 + 2x^2y + y^2 - y = C$

   i. $x + y\ln(x) = C$

Exam 1 Fall 2006

1. a. i. $\det(A - \lambda I) = \lambda^2 - 5\lambda + 6$

   ii. The eigenvalues of A are $\lambda = 2$ and $\lambda = 3$. $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector of A with eigenvalues $\lambda = 2$ and $\begin{pmatrix} 1/2 \\ 1 \end{pmatrix}$ is an eigenvector of A with eigenvalues $\lambda = 3$ and

   b. $\det(B - \lambda I) = -\lambda^3 + 2\lambda^2 - \lambda = -\lambda(\lambda - 1)^2$. The eigenvalues of B are $\lambda = 0$ and $\lambda = 1$.

2. $-\frac{1}{2}e^{-2y} = \frac{1}{3}e^{3x} + C$

3. a. $\frac{\partial z}{\partial x} = 4 \cdot \ln(xy) + 4$

   b. $\frac{\partial z}{\partial y} = \frac{4x}{y}$

4. $y_1(x) = 1 + x + \frac{1}{3}x^3$

   $y_2(x) = 1 + x + x^2 + \frac{2}{3}x^3 + \frac{1}{6}x^4 + \frac{2}{15}x^5 + \frac{1}{63}x^7$

5. The differential equation is exact since

$$\begin{aligned}\frac{\partial}{\partial y}\left(e^{2y} - y\cos(xy)\right) &= 2e^{2y} - \cos(xy) + xy\sin(xy) \\ &= \frac{\partial}{\partial x}\left(2xe^{2y} - x\cos(xy) + 2y\right)\end{aligned}$$

   The solutions are $xe^{2y} - \sin(xy) + y^2 = C$.

6. $\frac{y}{x} - 2\ln\left|\frac{y}{x} + 1\right| = \ln|x| + C$

7. $y^2 = \arctan(x) + 3$

8. $F(x) = x$ is an integrating factor for this equation since

$$\frac{\partial}{\partial y}\left(x(2y^2 + 3x)\right) = 4xy = \frac{\partial}{\partial x}\left(x(2xy)\right)$$

Exam 1 Spring 2007

1. $y = \dfrac{1}{x^3 + C}$

2. a. $\displaystyle\int \frac{e^x}{e^x+1}\,dx = \ln(e^x+1) + C$

   b. $\displaystyle\int \frac{dx}{x^2+2x+5} = \frac{1}{2}\arctan\left(\frac{x+1}{2}\right) + C$

   c. $\displaystyle\int \tan^2(x)\,dx = \tan(x) - x + C$

3. $y_1(x) = \dfrac{1}{2} + \dfrac{1}{4}x$

   $y_2(x) = \dfrac{1}{2} + \dfrac{1}{4}x - \dfrac{1}{48}x^3$

4. a. $\dfrac{\partial z}{\partial x} = y^2 e^{xy}$

   b. $\dfrac{\partial z}{\partial y} = e^{xy} + xye^{xy} - \tan(y)$

5. $y = \dfrac{2}{2 - e^x}$

6. The differential equation is exact since

   $$\frac{\partial}{\partial y}\left((x+1)e^x - e^y\right) = -e^y = \frac{\partial}{\partial x}\left(-xe^y\right)$$

   The solutions are $xe^x - xe^y = C$.

7. $\ln\left(\dfrac{y^2}{x^2} + 1\right) = -\ln|x| + C$ or $x^2 + y^2 = Cx$

8. $F(x) = x$ is an integrating factor since

   $$\frac{\partial}{\partial y}\left(x(2\cos(y) + 4x^2)\right) = -2x\sin(y) = \frac{\partial}{\partial x}\left(x(-x\sin(y))\right)$$

Exam 1 Fall 2007

1. i. $\det(A - \lambda I) = \lambda^2 - 6\lambda - 7$

   ii. The eigenvalues of A are $\lambda = 7$ and $\lambda = -1$. $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector of A with eigenvalues $\lambda = 7$ and $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$ is an eigenvector of A with eigenvalues $\lambda = -1$ and

2. $\tan(y) - y = \ln|x| - x^2 + C$

3. a. $\displaystyle\int \sin^3(x)\,dx = -\cos(x) + \frac{1}{3}\cos^3(x) + C$

   b. $\displaystyle\int \sin^2(x)\,dx = \frac{1}{2}x - \frac{1}{4}\sin(2x) + C$

4. $y_1(x) = 1 - x + x^2$

   $y_2(x) = 1 - x + 2x^2 - x^3 + \frac{1}{2}x^4 - \frac{1}{5}x^5$

5. The differential equation is exact since

$$\begin{aligned}\frac{\partial}{\partial y}\left(e^y + \cos(x)\cos(y)\right) &= e^y - \cos(x)\sin(y) \\ &= \frac{\partial}{\partial x}\left(-\left(3y^2 + \sin(x)\sin(y) - xe^y\right)\right)\end{aligned}$$

   The solutions are $xe^y + \sin(x)\cos(y) - y^3 = C$.

6. $\ln\left|\frac{y}{x}\right| - \ln\left|\frac{y}{x} + 1\right| = 2\ln|x| + C$ or $y = \dfrac{Cx^3}{1 - Cx^2}$

7. $y = \tan\left(x + \frac{\pi}{12}\right)$

8. $F(x) = \frac{1}{x}$ is an integrating factor since

$$\frac{\partial}{\partial y}\left(\frac{1}{x}(xy - 1)\right) = 1 = \frac{\partial}{\partial x}\left(\frac{1}{x}(x^2 - xy)\right)$$

Exam 1 Spring 2008

1. $x^2 + 5y^2 = C$

2. $\ln(e^y + 1) = -\ln(1 + \cos(x)) + 2\ln(2)$ which simplifies to $y = \ln\left(\dfrac{3 - \cos(x)}{1 + \cos(x)}\right)$

3. $F(x) = e^x$ is an integrating factor since
$$\frac{\partial}{\partial y}\left(e^x(y(x + y + 1))\right) = xe^x + e^x + 2ye^x = \frac{\partial}{\partial x}\left(e^x(x + 2y)\right)$$

4. $y_1 = 1 + x + \frac{1}{3}x^3$
   $y_2 = 1 + x + x^2 + \frac{2}{3}x^3 + \frac{1}{6}x^4 + \frac{2}{15}x^5 + \frac{1}{63}x^7$

5. $\ln\left|\frac{y}{x} + 1\right| + \ln\left|\frac{y}{x} - 1\right| = -\ln|x| + C$ or $y^2 = x^2 + Cx$

6. $y = \dfrac{C}{\sqrt{x^2 + 4}}$

7. $3\ln|x + 5| + 4\ln|x - 3| + C$

Exam 2 Review

1. $x^2 + 3y^2 = C$

2. a. $y(x) = -x^2 - 2cx - 2c^2 + Ce^{x/c}$

   b. $y(x) = \frac{1}{4}e^{3x} + Ce^{-x}$

   c. $x^4 + 2x^2y^2 - 2x^2 + y^4 + 2y^2 = C$

   d. $y(x) = \frac{1}{3} + Ce^{-x^3}$

   e. $y^3 = \frac{5x^6}{49 - 9x^5}$

   f. $y(x) = \frac{C}{1 + e^x}$

3. a. $i^{307} = -i$

   b. $e^{i\frac{\pi}{4}} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$

   c. $(1-i)^{400} = 2^{200} = 1606938044258990275541962092341162602522202993782792835301376$

4. a. The functions $\{\sin^2 x, 1 - \cos 2x\}$ are linearly dependent since
$$\frac{1 - \cos 2x}{\sin^2 x} = \frac{1 - (\cos^2 x - \sin^2 x)}{\sin^2 x} = \frac{1 - \cos^2 x + \sin^2 x}{\sin^2 x} = \frac{\sin^2 x + \sin^2 x}{\sin^2 x} = 2$$
   is constant.

   b. The functions $\{e^{2x}, e^{-2x}\}$ are linearly independent since
$$\frac{e^{-2x}}{e^{2x}} = e^{-4x}$$
   is not constant.

   c. The functions $\{\ln x^{-2}, \ln x^3\}$ are linearly dependent since
$$\frac{\ln x^3}{\ln x^{-2}} = \frac{3 \ln x}{-2 \ln x} = -\frac{3}{2}$$
   is constant.

5. a. $y_2(x) = e^{-5x}$

   b. $y_2(x) = e^{-x/2}$

6. a. $y(x) = c_1 e^{(-2+\sqrt{5})x} + c_2 e^{(-2-\sqrt{5})x}$

   b. $y(x) = 2\cos(4x) - \frac{1}{2}\sin(4x)$

   c. $y(x) = c_1 e^{2x} + c_2 x e^{2x} + c_3 x^2 e^{2x}$

7. a. $y(x) = 3x + c_1 + c_2 e^{-x}$

b. $y(x) = \frac{1}{7} x e^{4x} + c_1 e^{4x} + c_2 e^{-3x}$

c. $y(x) = -\frac{5}{36} e^{x} - \frac{1}{6} x e^{x} - \frac{1}{10} x - \frac{3}{100} + c_1 e^{2x} + c_2 e^{-5x}$

Exam 2 Fall 2006

1. $(1+i)^{200} = 2^{100} = 1267650600228229401496703205376$
2. The functions $\{\ln(x), x\ln(x)\}$ are linearly independent since $\dfrac{x\ln(x)}{\ln(x)} = x$ is not constant.
3. $y(x) = \dfrac{C}{\sqrt{x^2+9}}$
4. $2x^3 + 3x^2 + 3y^2 = C$
5. $y(x) = 3 - e^x + c_1 e^{3x} + c_2 e^{-x}$
6. $y(x) = \dfrac{x}{C + \ln|x|}$
7. $y(x) = 5e^x + 5xe^x$
8. $y_2(x) = x^{-3}$

Exam 2 Spring 2007

1. $\left(\frac{1+\sqrt{3}\,i}{2}\right)^{100} = -\frac{1+\sqrt{3}\,i}{2}$

2. The functions $\{\tan(x), \sec(x)\}$ are linearly independent since $\frac{\sec(x)}{\tan(x)} = \csc(x)$ is not constant.

3.

$f(x)$	Annihilator	y_p
$3x^3 + 6x - 2$	D^4	$a + bx + cx^2 + dx^3$
$3xe^{4x}$	$(D-4)^2$	$ae^{4x} + bxe^{4x}$
$\sin(5x)$	$D^2 + 25$	$a\cos(5x) + b\sin(5x)$
$e^{-2x}\cos(3x)$	$D^2 + 4D + 13$	$ae^{-2x}\cos(3x) + be^{-2x}\sin(3x)$
$\cos^2(x)$	$D(D^2+4)$	$a + b\cos(2x) + c\sin(2x)$

4. $y = \left(\frac{3}{4}x - c\right)^{2/3}$

5. $y(x) = \frac{5}{51}e^{6x} + c_1 e^{-x}\cos(\sqrt{2}\,x) + c_2 e^{-x}\sin(\sqrt{2}\,x)$

6. $y(x) = \frac{2e^x}{C - e^{2x}}$

7. $y(x) = 2e^{-6x} + 4xe^{-6x}$

8. $y_2(x) = x^{1/2}$

Exam 2 Fall 2007

1. $2x^2 + 5y^2 = C$

2.

$f(x)$	Annihilator	y_p
$x^2 e^{-x}$	$(D+1)^3$	$ae^{-x} + bxe^{-x} + cx^2e^{-x}$
$x^2 + e^{-x}$	$D^3(D+1)$	$a + bx + cx^2 + de^{-x}$
$\sin(x) \cdot \cos(x)$	$D^2 + 4$	$a\cos(2x) + b\sin(2x)$
$e^{3x}\cos(5x)$	$D^2 - 6D + 34$	$ae^{3x}\cos(5x) + be^{3x}\sin(5x)$
$x(1+e^{2x})$	$D^2(D-2)^2$	$a + bx + ce^{2x} + dxe^{2x}$

3. $y^{-2} = Cx^2 - x^4$

4. a.

$$\begin{aligned} x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y \\ &= x^2\left[\frac{3}{4}x^{-5/2}\cos(x) + x^{-3/2}\sin(x) - x^{-1/2}\cos(x)\right] \\ &\quad + x\left[-\frac{1}{2}x^{-3/2}\cos(x) - x^{-1/2}\sin(x)\right] \\ &\quad + \left(x^2 - \frac{1}{4}\right)x^{-1/2}\cos(x) \\ &= \frac{3}{4}x^{-1/2}\cos(x) + x^{1/2}\sin(x) - x^{3/2}\cos(x) \\ &\quad - \frac{1}{2}x^{-1/2}\cos(x) - x^{1/2}\sin(x) \\ &\quad + x^{3/2}\cos(x) - \frac{1}{4}x^{-1/2}\cos(x) \\ &= 0 \end{aligned}$$

b. $y_2(x) = x^{-1/2}\sin(x)$

5. $y(x) = -3e^{-7x} + 9xe^{-7x}$

6. The functions $\{\sin(x), x\sin(x)\}$ are linearly independent since $\dfrac{x\sin(x)}{\sin(x)} = x$ is not constant.

7. $y(x) = \dfrac{3}{25}e^{5x} + c_1e^x\cos(3x) + c_2e^x\sin(3x)$

Exam 2 Spring 2008

1. $y = \dfrac{x}{\ln|x| + C}$

2. $y = c_1 e^{-2x} + c_2 \sin(2x) + c_3 \cos(2x)$

3. $y = -1 + \sin(x) \ln|\sec(x) + \tan(x)| + c_1 \cos(x) + c_2 \sin(x)$

4. $y = \frac{1}{2} e^{-x^2} + 3e^{-2x^2}$

5. $y = c_1 x^3 + c_2 x^3 \ln(x)$

6. a. $(D + 5)^3$
   b. $D^2 + 2D + 5$
   c. D^6
   d. $y_p = A + Bx + Ce^{-x} + Dxe^{-x}$
   e. $y_p = Ae^x + Be^{-x/2} \cos(\sqrt{3}\, x/2) + Ce^{-x/2} \sin(\sqrt{3}\, x/2)$

7. a. $2x(x-1) \cdot 0 - (x+1) \cdot 1 + (1+x) = 0$
   b. $y_2 = -2x^{1/2}$

Exam 3 Review

1. (a) $y = e^x\cos(x)\ln|\cos(x)| + xe^x\sin(x) + c_1e^x\cos(x) + c_2e^x\sin(x)$

   (b) $y = \frac{1}{2}e^{8x} + \frac{1}{2}e^{-8x}$

   (c) $y = e^{-x}\ln(1+e^x) - e^{-x} + e^{-2x}\ln(1+e^x) + c_1e^{-x} + c_2e^{-2x}$

2. (a) $y = c_1x^{-1} + c_2x^2$

   (b) $y = c_1x^{1/2} + c_2x^{1/2}\ln(x)$

   (c) $y = c_1x^{-3} + c_2x^{-1}$

   (d) $y = c_1x^{-6} + c_2x^{-1}$

3. (a) $y = 16x^2 - 2x^4$

   (b) $y = 5x^2 - 7x^2\ln(x)$

4. After the substitution $x = e^t$ we have the differential equation
$$\frac{d^2y}{dt^2} - 5\frac{dy}{dt} + 6y = 2t$$
   Using the annihilator method we find that the general solution is
$$y(t) = \frac{5}{18} + \frac{1}{3}t + c_1e^{2t} + c_2e^{3t}$$
   Changing back to the variable x we get that the general solution to the original equation is
$$y(x) = \frac{5}{18} + \frac{1}{3}\ln(x) + c_1x^2 + c_2x^3$$

5. (a) $y_1 = c_1e^{-x} + c_2e^{5x}$, $y_2 = -c_1e^{-x} + 2c_2e^{5x}$

   (b) $y_1 = c_1e^{-4x} + c_2e^{4x}$, $y_2 = -2c_1e^{-4x} + 2c_2e^{4x}$

   (c) For any real numbers, b_1 and b_2, that satisfy the equation $2b_1 - 3b_2 = 1$,
$$\begin{aligned} y_1 &= 3c_1e^{6x} + 3c_2xe^{6x} + c_2b_1e^{6x} \\ y_2 &= 2c_1e^{6x} + 2c_2xe^{6x} + c_2b_2e^{6x} \end{aligned}$$
   is a form of the general solution. For instance, choosing $b_2 = 0$, $b_1 = 1/2$ and we get
$$\begin{aligned} y_1 &= 3c_1e^{6x} + 3c_2xe^{6x} + \tfrac{1}{2}c_2e^{6x} \\ y_2 &= 2c_1e^{6x} + 2c_2xe^{6x} \end{aligned}$$

   (d) $y_1 = 3e^{x/2}$, $y_2 = 2e^{-x/2} + 3e^{x/2}$

6. (a) $y = a_0\left(1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \cdots\right) + a_1\left(x + \frac{1}{6}x^3 + \frac{1}{120}x^5 + \cdots\right)$

   (b) $y = a_0\left(1 - \frac{1}{12}x^4 + \frac{1}{672}x^8 - \cdots\right) + a_1\left(x - \frac{1}{20}x^5 + \frac{1}{1440}x^9 - \cdots\right)$

   (c) $y = a_0\left(1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{7}{240}x^6 + \cdots\right) + a_1\left(x + \frac{1}{6}x^3 + \frac{1}{24}x^5 + \frac{1}{112}x^7 + \cdots\right)$

Exam 3 Fall 2006

1. $x^2 + y^2 = c$
2. $y = -\frac{1}{2}e^x \ln(1+x^2) + xe^x \arctan(x) + c_1 e^x + c_2 x e^x$
3. For any real numbers, b_1 and b_2, that satisfy the equation $3b_1 - b_2 = 1$,
$$\begin{aligned} y_1 &= c_1 + c_2 x + c_2 b_1 \\ y_2 &= 3c_1 + 3c_2 x + c_2 b_2 \end{aligned}$$
is a form of the general solution. For instance, choosing $b_1 = 0$, $b_2 = -1$ and we get
$$\begin{aligned} y_1 &= c_1 + c_2 x \\ y_2 &= 3c_1 + 3c_2 x - c_2 \end{aligned}$$
4. $c_1 = c_2 = -\frac{1}{2}$ and $\lim_{t\to\infty} q(t) = \dfrac{3}{2}$
5. $y(x) = c_1 x^{-2} + c_2 x^{-2} \ln(x)$
6. $c_1 = 1$ and $c_2 = 2$
7. No, this is not the solution.
8. $y = a_0 \left(1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{30}x^5 + \cdots\right) + a_1 \left(x + \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{120}x^5 + \cdots\right)$

Exam 3 Spring 2007

1. $y_p(x) = -\frac{1}{4}x^{-1}$

2. $y(x) = c_1x^{-3} + c_2x^{-3}\ln(x)$

3. $y_1(x) = 5c_1e^{8x} + c_2e^{-10x}$, $y_2(x) = 2c_1e^{8x} + 4c_2e^{-10x}$

4. p_1 and p_2 are any real numbers satisfying the equation $-5p_1 + 5p_2 = 1$. For instance, setting $p_1 = 0$ we have $p_2 = 1/5$.

5. $\sin(y) = \ln|\sec(x) + \tan(x)| + C$

6. (a) $(D-1)(D-3)$
   (b) $(D^2 + 4D + 29)^2$

7. $y = a_0(1 - x^2) + a_1\left(x - \frac{1}{6}x^3 - \frac{1}{120}x^5 - \cdots\right)$

Exam 3 Fall 2007

1. (a) $y_p(x) = Ae^{-3x}\cos(4x) + Be^{-3x}\sin(4x) + Cxe^{-3x}\cos(4x) + Dxe^{-3x}\sin(4x)$
   (b) $(D-3)(D+2)^2$
2. $\ln|\sec(y)| = \tan(x) + C$
3. $y_1 = c_1 + 2c_2e^{-5x}$, $y_2 = 3c_1 + c_2e^{-5x}$
4. $y(x) = c_1x^2 + c_2x^2\ln(x)$
5. $y(x) = \cos(\ln(x)) + 2\sin(\ln(x))$
6. $y_p = \frac{1}{2}x^3\ln(x) - \frac{3}{4}x^3$
7. $y = a_0\left(1 - \frac{1}{2}x^2 + \frac{5}{24}x^4 - \cdots\right) + a_1\left(x - \frac{1}{2}x^3 + \frac{7}{40}x^5 - \cdots\right)$

Exam 3 Spring 2008

1. a. $D^5(D-1)^4$
   b. $(D^2+9)^2$

2. $y = c_1 x\cos(2\ln(x)) + c_2 x\sin(2\ln(x))$

3. $y_p = -\frac{1}{4}x^{-1}$

4. $y_1 = c_1 e^{-x} + c_2 xe^{-x}$, $y_2 = c_1 e^{-x} + c_2 \frac{1}{5} e^{-x} + c_2 xe^{-x}$

5. $y = 3e^{-2x} + 11xe^{-2x}$

6. $q(t) = 10 - 10e^{-3t}\cos(3t) - 10e^{-3t}\sin(3t)$

7. $y = a_0(1 + x^2 + x^4 + x^6 + x^8 + \cdots) + a_1(x + x^3 + x^5 + x^7 + x^9 + \cdots)$

8. $y = \dfrac{a_0 + a_1 x}{1 - x^2}$

Problems on Page 125

1. $y = (t+1)e^{3t}$
2. $y = (t+2)e^{-4t}$
3. $\frac{2}{27} + \frac{1}{9}t - \frac{2}{27}e^{3t} + \frac{10}{9}te^{3t}$
4. $y = \frac{1}{20}t^5 e^{2t}$
5. $y = -\frac{3}{2}e^{3x}\sin(2x)$
6. $y = -e^{2t} + e^{3t} + u(t-1)\left(\frac{1}{6} - \frac{1}{2}e^{2t-2} + \frac{1}{3}e^{3t-3}\right)$
7. $y = 2e^{-t} + u(t-1)e^{-(t-1)}$
8. $y = u(t-2\pi)\frac{1}{4}\sin(4t)$

Problems on Page 126

1. $y = \frac{1}{2}x^2 + 2$
2. $y = 5e^{x-1}$
3. $y = x - 1 + e^{-x}$
4. $ye^x + y^2 = C$
5. $y = -2\cos^2(x) + 3\cos(x)$
6. $y^2 = \dfrac{1}{2} - x + Ce^{-2x}$
7. $y = x^4 + x^{-4}$
8. $y^2 = 1 + Ce^{-x^2}$
9. $y = 4e^{-3x} + 6e^{2x}$
10. $y = 3e^{2x} - 5xe^{2x}$
11. $y = c_1e^{-2x}\cos(x) + c_2e^{-2x}\sin(x)$
12. $y = c_1e^{-x/3} + c_2xe^{-x/3}$
13. $y = c_1e^{-x/5} + c_2e^{x/5}$
14. $y = c_1x^2 + c_2x^3$
15. $y = c_1x\cos(\ln(x)) + c_2x\sin(\ln(x))$
16. $y = 3x^{-1} - x^{-1}\ln(x)$
17. $y = 2\cos(3\ln(x))$
18. $y = 3 + \frac{1}{2}\sin(x) + c_1e^{-x/3} + c_2e^{-3x}$
19. $y = 4\cos(x) + 19\sin(x) + c_1e^{2x}\cos(4x) + c_2e^{2x}\sin(4x)$
20. $y = x\sin(3x) + \cos(3x)$
21. $y = \frac{1}{9}\ln|\cos(3x)|\cos(3x) + \frac{1}{3}x\sin(3x) + c_1\cos(3x) + c_2\sin(3x)$
22. $y = 2e^{-x}\sin(x)\tan(x) + c_1e^{-x}\cos(x) + c_2e^{-x}\sin(x)$
23. $y_1 = c_1e^{4x} + c_2e^{-3x}$, $y_2 = c_1e^{4x} - \frac{5}{2}c_2e^{-3x}$
24. $y_1 = c_1e^{-6x} + c_2xe^{-6x}$, $y_2 = -c_1e^{-6x} - c_2xe^{-6x} - \frac{1}{2}c_2e^{-6x}$
25. $y_1 = -2e^{4x} + 2e^{-3x}$, $y_2 = -2e^{4x} - 5e^{-3x}$
26. $y = C(x-2)^2e^x$

27. $y = \dfrac{C}{1-x^2}$

28. $y = a_0\left(1 + x^2 + \dfrac{1}{2}x^4 + \dfrac{1}{6}x^6 + \dfrac{1}{24}x^8 + \cdots\right) + a_1\left(x + x^3 + \dfrac{1}{2}x^5 + \dfrac{1}{6}x^7 + \dfrac{1}{24}x^9 + \cdots\right)$

$y = a_0 e^{x^2} + a_1 x e^{x^2}$

$a_n = \dfrac{(4n-6)a_{n-2} - 4a_{n-4}}{n(n+1)}$

29. $y = a_0\left(1 - \dfrac{1}{2}x + \dfrac{1}{12}x^2 - \dfrac{1}{144}x^3 + \dfrac{1}{2880}x^4 - \dfrac{1}{86400}x^5 + \cdots\right)$

$a_n = -\frac{1}{n(n+1)}a_{n-1}$

$a_n = \frac{(-1)^n}{(n+1)(n!)^2}a_0$

30. $y = 2e^t + e^{2t} - 2e^{3t} - u(t-2)(2e^t - 4e^{2t-2} + 2e^{3t-4})$

31. $y = 2\cos(4t) + \sin(4t)u(t-\pi)$