A FIBRATION FOR DIFF$\Sigma^n$

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0. Introduction

Let $M^n$ be an oriented smooth manifold, and let Diff $M^n$ be its group of orientation preserving diffeomorphisms. Let Diff$(N^n, M^n_0)$ and Diff$_* M^n$ be the two subgroups (with the $C^\infty$ topology) which consist of diffeomorphisms fixing $M^n_0 = M^n - D^n$ (for some embedded disc $D^n$) and the base point $* \in D^n \subset M^n$ respectively.

It is a common procedure when studying $\pi_\ast(Diff M^n)$ to restrict attention to either one of the above subgroups; it is therefore of some interest to study the homotopy braid of the triple $(Diff M^n, Diff_* M^n, Diff(M^n, M^n_0))$.

In fact this is equivalent to considering the following ‘comparison diagram’ of principal fibrations, constructed by the usual techniques.

(0.1) Diagram.

\[
\begin{array}{cccccc}
\text{Diff}_* M^n & \xrightarrow{d} & \text{SO}(n) \\
\downarrow j' & & \downarrow j \\
\text{Diff}(M^n, M^n_0) & \xrightarrow{i} & \text{Diff} M^n & \xrightarrow{e} & P\tau(M^n) \\
\downarrow i & & \downarrow \pi & & \downarrow \pi \\
M^n & & & & \\
\end{array}
\]

Here $d$ is the derivative map at $*$, $P\tau(M^n)$ is the total space of the oriented smooth principal tangent bundle to $M^n$, $e$ maps $f \in \text{Diff} M^n$ to the derivative at $*$ of the restriction $f|D^n$, and $\pi$ is evaluation at $*$.

The only fibration on display above which is not well documented is the lower horizontal one, i.e.

\[
\text{Diff}(M^n, M^n_0) \xrightarrow{i} \text{Diff} M^n \xrightarrow{e} P\tau(M^n)
\]

(F)

Its existence follows from the fibration

\[
\text{Diff}(M^n, M^n_0) \xrightarrow{i} \text{Diff} M^n \xrightarrow{e'} \mathcal{E}(D^n, M^n),
\]

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where \( \mathcal{E}(\cdot, \cdot) \) denotes the space of orientation preserving smooth embeddings on \( D^n \). But 
\( \mathcal{E}(D^n, M^n) \) may be identified with \( P\tau(M^n) \), since
\[
\mathcal{E}_*(D^n, M^n) \to \mathcal{E}(D^n, M^n) \to M^n
\]
is a model for the principal tangent bundle of \( M^n \) via the derivative map.

The homotopy braid of (0.1) gives rise to two interlaced problems. Firstly, to what extent does the ‘linearization’ \( j_P \) determine the map \( j \); and secondly, to what extent does \( d \) determine \( e \)?

We shall discuss the second of these questions in the special case of \( M^n \) an exotic sphere \( \Sigma^n \). In this case, (F) generalizes an exact sequence described by R. Schultz [6]. In particular we study the difference between the boundary maps associated to \( d \) and to \( e \), and reduce the detection of a certain class of ‘stable’ homotopy element so arising to an interesting, but apparently unsolved, problem in the homotopy groups of spheres.

Throughout, we shall write \( S^n_\beta \) for the exotic sphere given by an element \( \beta \in \pi_n(\text{Top}/\text{O}) \), \( n \geq 7 \). Such \( \beta \) arises from an isotopy class of diffeomorphisms \( \beta \in \text{Diff}S^{n-1}_\beta \), so any \( S^n_\beta \) can be represented as \( D^n_0 \cup_\beta D^n_\beta \), where \( 0 \in D^n_0 \) is the basepoint, and \( D^n_\beta \) is the complementary disc.

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1. The fibration

Our fibration (F) of §0 can be further simplified when \( M^n = S^n_\beta \). For it has long been known that, whatever \( \beta \), \( P\tau(S^n) = \text{SO}(n+1) \). It is most convenient to describe this fact by means of

(1.1). Lemma. There is a homeomorphism of degree 1, say \( h : S^n_\beta \to S^n \), such that the diagram

\[
\begin{array}{ccc}
S^n_\beta & \to & S^n \\
\downarrow h & & \downarrow \tau(S^n) \\
\tau(S^n_\beta) & \to & B\text{SO}(n)
\end{array}
\]

homotopy commutes. Thus \( P\tau(S^n_\beta) \) is homeomorphic to \( \text{SO}(n+1) \) via an \( \text{SO}(n) \) equivariant map,

Proof. Since \( S^n_\beta \) is stably parallelizable, \( \tau(S^n_\beta) \) lifts to \( S^n \). We can choose the lift \( h \) to have degree 1 by appealing to the euler characteristic if \( n \) is even, and the Kervaire semicharacteristic if \( n \) is odd. \( \square \)
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(1.2). **Note.** The resulting homeomorphism $\bar{h} : P\tau(S^n) \to \text{SO}(n + 1)$ is defined only up to alteration by any map $S^n_\beta \to \text{SO}(n)$.

**Proof.** Since $S^n_\beta$ is stably parallelizable, $\tau(S^n_\beta)$ lifts to $S^n$. We can choose the lift $h$ to have degree 1 by appealing to the euler characteristic if $n$ is even, and the Kervaire semi-characteristic if $n$ is odd. $\square$

We can now construct our special version of (0.1) as follows:

(1.3). **Diagram.**

$$
\begin{array}{c}
\text{Diff}_* S^n_\beta \\
\downarrow i' \\
\text{Diff}(D^n, \partial) \downarrow j \downarrow j_P \downarrow B \text{Diff}(D^n, \partial) \\
\text{Diff}_* S^n_\beta \downarrow \varepsilon \downarrow \text{SO}(n) \\
\downarrow \pi \downarrow \text{SO}(n + 1) \\
\downarrow \pi \\
S^n \\
\end{array}
$$

Note that we have labeled $\bar{h} \circ e$ as $\varepsilon$, and the classifying map of $\varepsilon$ as $g(\beta)$. Of course $j_P$ is precisely the standard inclusion, and the homotopy commutativity of the central square is assured by construction.

(1.4). **Definitions.**

(i) Let $W_*(S^n) \subset \pi_* (\text{Diff } S^n)$ be the graded subgroup $\text{Im } i' \cap \ker j_*$. 

(ii) Let $X_*(S^n) \subset \pi_* (\text{SO}(m + 1))$ be the graded set of elements $x$ with the property that $\pi_* (x) \neq 0 \neq g(\beta)_* (x)$.

Thus $W_*(S^n_\beta)$ is a measure of the extent to which $j_P$ fails to determine $j$, and $X_*(S^n_\beta)$ is a measure of the extent to which $\partial'$ fails to determine $g(\beta)$. Also $0 \neq x \in W_*(S^n_\beta)$ yields $(i')^{-1}(w) \in X_*(S^n_\beta)$.

We investigate $X_*(S^n_\beta)$ below. Note that, if $k < n$, then $W_k(S^n_\beta) = 0$ and $X_k(S^n_\beta) = \emptyset$. Also, $W_*(\ )$ and $X_*(\ )$ are defined for arbitrary $M^n$.

If $S^n_\beta$ is the standard sphere $S^n$, then (1.3) ‘collapses’. For the symmetry of $S^n$ allows splitting $\text{Diff } S^n \leftarrow \text{SO}(n + 1)$ of (F), which restricts to a splitting of the upper fibration. Thus $W_*(S^n) = 0$ and $X_*(S^n) = \emptyset$.

Hence the cardinalities of $W_*(\ )$ and $X_*(\ )$ in some sense reflect the asymmetry of $S^n_\beta$. We develop below a detection procedure for ‘stable’ elements in $X_*(\ )$. 

(1.5).
2. Detecting elements in $X_*(\ )$

We first summarize some information from [6] concerning the map $\partial'$ of (1.3). Before so doing, however, it is convenient to recall some familiar notation which will also be useful for the remainder of this section.

We shall write $\text{Top} S^n$ for the group of orientation preserving homeomorphisms of $S^n$, so that we have the following commutative diagram:

\[
\begin{array}{ccc}
\text{SO} & \to & S \text{Top} \\
\downarrow & & \downarrow \text{Top} / O \\
\text{SO}(n + 1) & \overset{i}{\to} & S \text{Top}(n) \\
\downarrow & & \downarrow k \\
\text{SO} & \overset{j}{\to} & \Omega^{n+1} S^{n+1}
\end{array}
\]

Here $i$ is the standard inclusion, $j$ is the suspension, and $k$ is one-point compactification.

Now according to [1], the composite $\pi_* (S \text{Top}(n)) \to \pi_* (\text{Top} / O)$ is epic, so we may suppose that $S^n$ is represented by a map $S^n \to S \text{Top}(n)$ (tantamount to choosing a framing for $S^n$); or, via $k_\ast$, a map $\tilde{\beta} : S^n \to \text{Top} S^n$. Continuing around the diagram, we also obtain $\tilde{\beta} \in \pi_n(G/O)$ as the image of both $\tilde{\beta}$ and $\beta$. Moreover $\tilde{\beta}$ lies in the summand $\pi_n(G)/\text{Im} J$.

We can now state

(2.1). Theorem. (R. Schultz). Given $\alpha \in \pi_k(\text{SO}(n))$, then $\partial' \alpha \neq 0$ in $\pi_k(B \text{Diff}(D^n, \partial))$ whenever $\tilde{\beta} J(\alpha) \neq 0$ in $\pi^S_{n+k}/\text{Im} J$.

Our result concerning $X_*(\ )$ is in the same spirit, and can be stated thus:

(2.2). Theorem. Given $x \in \pi(\text{SO}(n + 1))$, then $x \in X_k(S^n)$ whenever $\tilde{\beta} J(\alpha) \neq 0$ in $\pi^S_{n+k}/\text{Im} J$.

Here $H(x) \in \pi_k(S^n)$ is represented by the composite $S^k \overset{\pi}{\to} \text{SO}(n + 1) \overset{\pi}{\to} S^n$ ($\pi$ being the usual projection), which is proven in [2] to be the Hopf invariant (up to sign and suspension) of the element $J(x) \in \pi_{k+n+1}(S^{n+1})$.

Elements so detectable constitute a stable subset $SX_*(S^n) \subset X_*(S^n)$. Unfortunately, we know of no non-zero classes of the form $y^2 \cdot H(x)$ in any stem mod $\text{Im} J$. The experts seem to
regard this as a potentially accessible, but unsolved, problem of homotopy theory. Certainly, all \( x \in \pi_{n+1}(SO(n+1)) \) give \( y^2 \cdot H(x) = 0 \) for all \( y \) when \( t \) is small.

There are two main steps involved in establishing (2.2): these follow below as (2.3) and (2.4).

The first includes generalizing a diagram of [6, p. 240]. Since in (1.3) we have set up a map \( \varepsilon : \text{Diff}_S^n \to SO(n+1) \) (which depends on the choice of \( \bar{h} \) in (1.2)), it is important to relate \( \varepsilon \) with the standard inclusion of both \( \text{Diff}_S^n \) and \( SO(n+1) \) in \( \text{Top} S^n \). This is done by

(2.3). Lemma. The following diagram is homotopy commutative:

\[
\begin{array}{ccc}
\text{Diff}_S^n & \xrightarrow{\varepsilon} & \text{Top} S^n \\
\downarrow{\bar{h}} & & \downarrow{c_\beta} \\
SO(n+1) & \xrightarrow{i} & \text{Top} S^n \\
\end{array}
\]

Here \( \chi_h \) is conjugation by the homeomorphism \( h \), whereas \( c_\beta \) is the composition

\[
\text{Top} S^n \xrightarrow{\pi \times 1} S^n \times \text{Top} S^n \xrightarrow{\beta \times 1} \text{Top} S^n \times \text{Top} S^n \rightarrow \text{Top} S^n
\]

where \( \pi \) projects a homeomorphism onto its value at \( 0 \in S^n = \mathbb{R}^n \cup \{\infty\} \), and \( \mu \) is composition of functions. Note that \( \pi \circ \beta (f) = \pi(f) \in S^n \) for all \( f \), and that \( c_\beta \) is a homeomorphism with inverse \( c_{-\beta} \).

The proof of (2.3) proceeds by passing between three equivalent versions of \( P^{\text{Top} \tau}(S^n) \), the oriented principal topological tangent bundle of \( S^n \). These may be displayed by the commutative diagram of (principal) fibrations

\[
\begin{array}{ccc}
S \text{Top}(n) & \xrightarrow{\sigma_1} & \text{Top}_* S^n \\
\downarrow{\pi} & & \downarrow{1} \\
P^{\text{Top} \tau}(S^n) & \xrightarrow{\varphi_1} & \text{Top} S^n
\end{array}
\]

The maps \( \sigma_1 \) and \( \varphi_1 \) are induced by compactification, and \( \sigma_2 \) and \( \varphi_2 \) be restricting a homeomorphism of \( \mathbb{R}^n \cup \{\infty\} \) to \( D^n \).

To introduce the second ingredient in the proof of (2.2), let us return to our fibration of (1.3). Suppose that \( \alpha : Y \to SO(n+1) \) is a map of some reasonable space into the base. If \( \alpha \) does not factor through \( \varepsilon : \text{Diff}_S^n \to SO(n+1) \) and does not lift to \( SO(n) \), then it
represents a class with the properties we are seeking for $X_*(S^n_\beta)$ (in case $Y$ is a sphere). We thus wish to discuss the obstruction to lifting $\alpha$ to $\text{Diff} S^n_\beta$.

We may assume without loss of generality that the suspension $S^1 \wedge Y$ is given as an open subset of some euclidean space $\mathbb{R}^n$; in other words as an open smooth manifold. Now let $\alpha^* \gamma_\beta$ be the topological $S^n$ bundle over $S^1 \wedge Y$ which arises by adjoining the composite

$$Y \xrightarrow{\alpha} \text{SO}(n+1) \xrightarrow{i} \text{Top} S^n \xrightarrow{\gamma_\beta} \text{Top} S^n \xrightarrow{\epsilon_\beta} \top S^n$$

Then if $\alpha$ lifts to $\text{Diff} S^n_\beta$, (2.3) tells us that the total space $E(\alpha^* \gamma_\beta)$ admits a smoothing which restricts to $\beta$ on each fibre.

In fact it is most useful to work universally, and to consider the case of $Y = \text{SO}(n+1)$ and $\alpha$ the identity map. Then $E(\gamma_\beta)$ can be constructed by first choosing the ‘core’ plus a single fibre, i.e. $S^n \cup C \text{SO}(n+1)$, where the attaching map is $\pi \circ \gamma_\beta = \pi$. To this we must further attach a cone on the join $S^{n-1} \ast \text{SO}(n+1)$ by a suitable map $\eta$. We therefore have a cofibre sequence

$$S^n \wedge \text{SO}(n+1) \xrightarrow{\eta} S^n \cup_\pi C \text{SO}(n+1) \xrightarrow{\theta} E(\gamma_\beta)$$

But $E(\gamma_\beta)$ is a topological manifold, fibered by $S^n$’s and over a smooth base. As such it admits a $\text{STop}(n)$ bundle of tangents along the fibres, say

$$\text{Top}_{\tau_F} : E(\gamma_\beta) \rightarrow B \text{STop}(n).$$

Our aim is to determine the extent to which $E(\gamma_\beta)$ admits a smoothing fibred by $S^n$’s, or equivalently to which it carries an $n$-plane bundle $\tau_F$, agreeing with $\text{Top}_{\tau_F}$ topologically and restricting to $\tau(S^n_\beta)$ on each fibre.

Now from §1, $\tau_\beta : S^n_\beta \rightarrow B \text{SO}(n)$ extends to some bundle $\tau_\beta$ over $S^n_\beta \cup C \text{SO}(n+1)$. Thus we may construct $\tau_F$ at least over $S^n \cup_\pi C \text{SO}(n+1)$, by composing $\tau_\beta$ with $h^{-1}$.

Returning to our cofibration (C), we can consider $\eta^* \tau_F$ over $S^n \wedge \text{SO}(n+1)$. This is topologically trivialized by the existence of $\text{Top}_{\tau_F}$, so we have a map $\sigma(\beta) : S^n \wedge \text{SO}(n+1) \rightarrow \text{Top}(n)/\text{O}(n)$ which fits into the following homotopy commutative diagram

$$S^n \wedge \text{SO}(n+1) \xrightarrow{\eta} S^n \cup_\pi C \text{SO}(n+1) \xrightarrow{\theta} E(\gamma_\beta) \xrightarrow{\tau_F} \text{Top}(n)/\text{O}(n) \xrightarrow{\sigma(\beta)} BS \text{SO}(n) \xrightarrow{\text{Top}_{\tau_F}} BS \text{Top}(n)$$

So $\sigma(\beta)$, which we shall confuse with its adjoint $\text{SO}(n+1) \rightarrow \Omega(\text{Top}(n)/\text{O}(n))$, is the obstruction to extending $\tau_F$ over the whole of $E(\gamma_\beta)$. 
Thus in terms of our original $\alpha^*\gamma_\beta$ we deduce that if $\alpha : Y \to \text{SO}(n+1)$ lifts to $\text{Diff} S^\beta_\beta$ then the composite

$$Y \xrightarrow{\alpha} \text{SO}(n+1) \xrightarrow{\sigma(\beta)} \Omega^n(\text{Top}(n)/\text{O}(n))$$

is null-homotopic.

We have shown only that this map is a necessary obstruction to lifting $\alpha$. In the light of the celebrated Morlet equivalence $B \text{Diff}(D^n, \partial) \simeq \Omega^n(\text{Top}(n)/\text{O}(n))$ (e.g. see [3]) it seems highly likely that $\sigma(\beta)$ and $g(\beta)$ of (1.3) are the same map. Note that on homotopy groups $\sigma(\beta)$ induces a non-bilinear extension to $\pi_k(\text{SO}(n+1))$ of the Milnor pairing $\langle \cdot \beta \rangle : \pi_k(\text{SO}(n)) \to \pi_{k+n}(\text{Top} / \text{O})$.

For calculational purposes, and given the current state of the art, unstable results such as we have obtained are not especially helpful. We must therefore show

(2.4). Lemma. The stabilization

$$\sigma(\beta) : \text{SO}(n+1) \xrightarrow{\sigma(\beta)} \Omega^n(\text{Top}(n)/\text{O}(n)) \to \Omega^n(\text{Top} / \text{O})$$

may be described as

$$\text{SO}(n+1) \xrightarrow{\tau_\beta} \text{Top} S^n \xrightarrow{J} \Omega^{n+1} S^{n+1} \xrightarrow{\pi_{n+1}} \Omega^{n+1} B(\text{Top} / \text{O})$$

This formula follows simply from stabilizing the bundles in our discussion above.

To complete the proof of (2.2), we must choose $Y = S^k$ and $\alpha$ to represent a class $x \in \pi_k(\text{SO}(n+1))$ such that $\pi_*(x) \neq 0$ in $\pi_k(S^n)$. Then by (2.4), $x \in X_\beta(S^n)$ if

$$S^k \xrightarrow{x} \text{SO}(n+1) \xrightarrow{\sigma(\beta)} \Omega^n(\text{Top} / \text{O})$$

is not null-homotopic. The usual detection procedure for such a map is then to pass to $\Omega^n(G/O)$, and to compute its value in the summand $\pi^{S^k}_{n+k}/\text{Im} J \subset \pi_{k+n}(G/O)$.

In our case the maps involved can be unraveled to give $\sigma(\beta)_x$ module $\text{Im} J$ as

$$S^k \xrightarrow{x} \text{SO}(n+1) \xrightarrow{\pi \times J} S^n \times G \xrightarrow{-\beta \times 1} G \times G \xrightarrow{\circ} G \xrightarrow{\beta} \Omega^n G.$$

This represents $\tilde{\beta} \circ (J(x) - \beta(\pi \circ x))$ in $\pi^S_{n+k}/\text{Im} J$. But $\pi \circ x = H(x)$, whilst Novikov [5] and Kosinski [4] have shown that $\tilde{\beta} \circ J(x) \in \text{Im} J$ whenever $k > \frac{1}{2}n + 1$ (which is certainly the case here).

We can now deduce our detection formula (2.2) in the form

$$\sigma(\beta)_x x = \pm \tilde{\beta}^2 : H(x) \quad \text{in} \quad \pi^{S^k}_{n+k}/\text{Im} J.$$

Note that if $S^\beta_\beta$ bounds a parallelizable manifold, then $\tilde{\beta} = 0$ by definition. So $SX_*(S^\beta_\beta) = \emptyset$. We conclude with a result which is a more subtle version of this same fact

(2.5). Proposition. Let $S'X_*(S^\beta_\beta)$ be the intermediate set $SX_*(S^\beta_\beta) \subset S'X_*(S^\beta_\beta) \subset X_*(S^\beta_\beta)$ of elements detected by $\sigma(\beta)_x x \in \pi_{n+k}(\text{Top} / \text{O})$. Then $S'X_*(S^\beta_\beta) = \emptyset$ if $S^\beta_\beta$ bounds a parallelizable manifold.
Proof. By choice, $\beta \in \pi_{n+k}(B\text{Top}/O)$ lifts to $\pi_n(G/\text{Top})$. But localized at 2, $G/\text{Top}$ is a product of Eilenberg-MacLane spaces, and at odd primes is equivalent to $BO$. In either case $\beta \circ f = 0$ for any $f \in \pi_{k+n+1}(S^{n+1})$.

This may be one more way of saying that such $S^n_\beta$’s are the most symmetric of exotic spheres.

References


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