SMOOTHING H-SPACES II

ERIK KJÆR PEDERSEN

0. Introduction

In [8] we showed that certain $H$-spaces obtained by homotopy mixing are homotopy equivalent to smooth, parallelizable manifolds. Unfortunately (as was added in proof) we needed the restriction on the fundamental group $\pi$, that $D(Z\pi) = 0$. It is the purpose of this sequel to [8] to remove this restriction, and also to generalize the main theorem considerably. I want to thank I. Hambleton for pointing out the error in [8].

1. Notation. Statement of results

Throughout the paper space will mean topological space of the homotopy type of a connected CW complex. For a set of primes $l$ and a nilpotent space $X$, $X_l$ denotes the localization at $l$ in the sense of [5]. A space $X$ is called quasifinite if $H_*(X) = \oplus H_i(X; Z)$ is a finitely generated abelian group. If $H_*(X)$ is a $Z_l$-module $X$ is called $l$-locally quasifinite if $H_*(X)$ is finitely generated as a $Z_l$-module.

To state our main theorem we need a couple of definitions. Let $S^i$ denote the $i$-sphere.

1.1. Definition. A nilpotent space $X$ admits a special 1-torus if, up to homotopy, there is a diagram of orientable fibrations

\[
\begin{array}{ccccccccc}
S^1 & \to & S^3 & \to & S^2 \\
\downarrow & & \downarrow & & \downarrow \\
S^1 & \to & X & \to & B \\
\downarrow & & \downarrow & & \downarrow \\
* & \to & A & \to & A
\end{array}
\]

such that

(a) $A$ is quasifinite, $B$ is stably reducible.
(b) Localized at 0 the diagram is homotopy equivalent to
We remark that since $X$ is nilpotent and the fibrations are orientable $A$ and $B$ are nilpotent so localization makes sense.

1.2. Example. Given a bundle $S^3 \to X \to A$ with $A$ quasifinite stably reducible and the bundle stably trivial then $X$ admits a special $1$-torus by dividing out the subgroup $S^1 \subset S^3$ (see [8, lemma 3.6]). All compact Lie groups other than $SO(3)^k \times T^l$ have subgroups isomorphic to $S^3$ as is seen by classification and all these Lie groups admit special $1$-tori. This will be further discussed in section 5.

We need a $p$-local version of definition 1.1. Let $X$ be a nilpotent space, $p$ a prime.

1.3. Definition. $X$ admits a $p$-local special $1$-torus if, up to homotopy, there is a diagram of orientable fibrations

\[
\begin{array}{cccccc}
S^1_p & \to & S^3_p & \to & S^2_p \\
\downarrow & & \downarrow & & \downarrow \\
S^1_p & \to & A_0 \times S^3_p & \to & A_0 \times S^2_p \\
\downarrow_{pr_1} & & \downarrow_{pr_1} & & \\
* & \to & A_0 & \to & A
\end{array}
\]

such that

(a) $A$ is $p$-locally quasifinite and $B$ is $p$-locally stably reducible, i.e. there are integers $n$ and $i$ and a map $S^{n+i}_p \to \Sigma^1 B$ inducing isomorphism in homology in dimensions $\geq n + i$.

(b) as in definition 1.1

It is clear that if $X$ admits a special $1$-torus then $X_p$ admits a $p$-local special $1$-torus.

We prove the following

1.4. Theorem. Let $X$ be a quasifinite $H$-space. Assume for every prime $p$ that $X_p$ is homotopy equivalent to a product $C(p) \times D(p)$ and $C(p)$ admits a $p$-local special $1$-torus. Then $X$ is homotopy equivalent to a smooth, stably parallellizable manifold.

1.5. Remark. If $H_3(X) \subseteq \mathbb{Z}$ the condition of the theorem is trivially satisfied for all but finitely many primes. This is because for all but finitely many primes $X_p$ is homotopy
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equivalent to a product of localized spheres which must include the 3-sphere. We also note that in view of example 1.2 (see section 5) this theorem is stronger than theorem 1.1 of [8].

2. Surgery

2.1. Proposition. Let $X$ be a quasifinite, nilpotent Poincaré complex admitting a special 1-torus. Then $X$ is homotopy equivalent to a stably parallelizable smooth manifold.

Remark. This result only needs condition (a) of definition 1.1 Condition (b) is needed to ensure that the property of having a special 1-torus is a generic property.

Proof of proposition 2.1. In the diagram of orientable fibrations

$$
\begin{array}{ccc}
S^1 & \rightarrow & S^3 \\
& \downarrow & \downarrow \\
S^1 & \rightarrow & X \\
& \downarrow & \downarrow \\
& \rightarrow & B \\
& \downarrow & \downarrow \\
& \rightarrow & A \\
& \downarrow & \downarrow \\
* & \rightarrow & A
\end{array}
$$

A is nilpotent and quasifinite, hence by [7] $A$ is finitely dominated. It follows that $X$ and $B$ are finitely dominated [6]. Also $A$ and $B$ are Poincaré Duality spaces since $X$ is [3]. It follows from [10] that $X$ has 0 finiteness obstruction. Considering $(B, X)$ a Poincaré Duality pair, we may use the stable reduction of $B$ and a standard transversality procedure to produce a surgery problem

$$(M, \partial M) \xrightarrow{\phi} (B, X) \xrightarrow{\phi : \nu_M} \varepsilon$$

where $\varepsilon$ is the trivial bundle. Let $\sigma(B)$ be the finiteness obstruction of $B$. Consider the exact sequence

$$\ldots \rightarrow H^{n+1}(\mathbb{Z}_2; \tilde{K}_0(\mathbb{Z}_\pi)) \rightarrow L^h_n(\pi) \xrightarrow{\delta} L^p_n \rightarrow$$

where $\pi = \pi_1(B) = \pi_1(X)$. The class of $\sigma(B)$, $\{\sigma(B)\}$, is an element of $H^{n+1}(\mathbb{Z}_2; \tilde{K}_0(\mathbb{Z}_\pi))$. It follows from [9], that the surgery obstruction of $\partial M \rightarrow X$ is $\delta\{\sigma(B)\}$. However, since $A$ is a P. D, space of dimension $n - 3$ we have $\sigma(A) = (-1)^{n-3}\sigma(A)^*$ and by [10], $\sigma(B) = 2\sigma(A) = \sigma(A) + (-1)^{n+1}\sigma(A)^*$ and hence $\{\sigma(B)\} = 0$ in $H^{n+1}(\mathbb{Z}_2; \tilde{K}_0(\mathbb{Z}_\pi))$ and we are done.


Browder and Spanier have shown that a finite $H$-space is stably reducible [2]. This is one of the steps in the attempt to prove $X$ is a manifold, since it implies that the Spivak normal fibre space is trivial. We need to generalize the results of Browder and Spanier to a $p$-local situation. This is mostly straightforward. We shall nevertheless indicate the line of argument in this section. The aim of this section is to prove:
3.1. **Theorem.** Let $D$ be a $p$-locally quasi finite $H$-space. Then $D$ is $p$-locally stably reducible.

We need a $p$-local edition of $S$-duality.

3.2. **Proposition.** Let $X$ be a simply connected $p$-locally quasifinite space. Then $X$ admits a $p$-local CW-structure, i.e. $X$ is homotopy equivalent to a space $Y$ with a filtration $* = K_0 \subset K_1 \subset \ldots \subset K_n = Y$ such that $K_i$ is the mapping cone of some map $f_i : S^{n(i)}_p \to K_{i-1}$, $n(i)$ a nondecreasing function of $i$.

**Proof.** By the Hurewicz theorem we may find a finite wedge of local spheres and a map $f \vee S^n_p \to X$ such that $H_*(f)$ is onto in dimensions $\leq k$, $k \leq 2$. Using the relative Hurewicz theorem we inductively attach local cells to make $H_*(f)$ an isomorphism in higher dimensions.

Since $X$ is $p$-locally quasifinite and finitely generated $\mathbb{Z}_p$-module have free resolutions of length one, we eventually obtain a homotopy equivalence.

Let $X$ and $Y$ be $p$-locally quasifinite spaces. A $p$-local $S$-duality map is a map $X \wedge Y \to S^n_p$ so that slant product $f^*(i)/- : \tilde{H}_*(X) \to \tilde{H}^{n-*}(Y)$ is an isomorphism. Here $i$ is the generator of $H^n(S^n_p)$.

Given a $p$-locally finite space $X$ we note that the suspension $\Sigma X$ is a simply connected $p$-locally quasifinite space and thus admits a $p$-local CW structure by Proposition 3.2. We may now go through exercises F1-7 page 463 in Spanier [11] to prove existence and stable uniqueness of a $p$-local $S$-dual with the usual functorial properties. We need the concept to complete the proof of Theorem 3.1.

**Proof of Theorem 3.1.** $H^*(D; \mathbb{Q})$ and $H^*(D; \mathbb{Z}/p\mathbb{Z})$ are Hopf algebras and we may argue as in the finite case [1] that $D$ satisfies Poincaré Duality with $\mathbb{Z}_p$ coefficients. We now only need to produce a map $D \to S^n_p$ inducing isomorphisms in dimensions $\geq n$. Then we may use Hopf algebra arguments (as in the finite case [2]) to prove that the composite $D^+ \wedge D^+ = (D \times D)^+ \to D \to S^n_p$ is a $p$-local $S$-duality map, so $D^+$ is selfdual and the dual of $D^+ \to S^n_p$ will be a stable reduction.

Localized at $0$ $D$ is a product of odd dimensional spheres so if we let $l$ be the set of primes different from $p$ and form the homotopy pullback

\[
\begin{array}{ccc}
Y & \longrightarrow & \prod S^{2n+1}_i \\
\downarrow & & \downarrow \\
D & \longrightarrow & \prod S^{2n+1}_0 
\end{array}
\]

then $Y$ is quasifinite, nilpotent and satisfies Poincaré Duality at all primes hence [5] and [7] is a finitely dominated Poincaré Duality space. By Wall [12] $Y$ has the homotopy type of $K \cup e^n$ where $K$ is $n-1$-dimensional and we may thus produce a map $Y \to K \cup e^n \to S^n$ by
collapsing $K$ to a point. Localizing at $p$ we obtain $D \cong K_p \to S^n_p$ with the required property and we are done

4. PROOF OF MAIN THEOREM.

The proof will consist of two lemmas.

4.1. LEMMA. If $X$ is a quasifinite $H$-space and $X_p = C(p) \times D(p)$ where $C(p)$ admits a $p$-local special 1-torus, then $X_p$ does.

Proof. Crossing the special 1-torus diagram

$$
\begin{array}{c}
C(p) \to B \\
\downarrow \downarrow \\
A \to A
\end{array}
$$

with $D(p)$ reduces the lemma to showing $B \times D(p)$ is $p$-locally stably reducible. Now $D(p)$ is a retract of a $p$-locally quasifinite $H$-space and is thus itself a $p$-locally quasifinite $H$-space and thus $p$-locally stably reducible by theorem 3.1

4.2. LEMMA. If $X$ is a quasifinite $H$-space such that each $X_p$ admits a $p$-local special 1-torus, then $X$ admits a special 1-torus.

Proof. At all but finitely many primes $X$ is a product of spheres, so we may consider $X$ a homotopy pullback

$$
\begin{array}{c}
X_i \leftarrow X_{p_1} \leftarrow \cdots \leftarrow X_{p_k} \\
\downarrow \downarrow \downarrow \downarrow \\
X_0
\end{array}
$$

where $X_i$ is a product of odd dimensional spheres and $X_{p_i}$ also admit special 1-tori. Mixing the special 1-tori in the obvious way we obtain $\overline{X}$ which admits a special 1-torus and such that $\overline{X}_i \cong X_i$ and $\overline{X}_{p_i} \cong X_{p_i}$; in other words $\overline{X}$ is in the genus of $X$. We now argue as in [8, proposition 3.2] to show that admitting a special 1-torus is a generic property for an $H$-space. The key step is the result of Zabrodsky that one obtains the whole genus of an $H$-space by mixings defined by diagonal matrices and the observation in [8] that one of these diagonal entries may be assumed to be 1.
5. Examples.

In this section we show that compact Lie groups other than $SO(3)^k \times T^l$ do admit special 1-tori. This implies that our theorem 1.4 is indeed stronger than theorem 1.1 of [8].

5.1. Proposition. Let $G$ be a compact connected Lie group which is not isomorphic to $SO(3)^k \times T^l$. Then $G$ has a subgroup isomorphic to $S^3$.

Proof. We use classification of compact Lie groups. Any compact connected Lie group is a quotient of $H \times T^l$ by a discrete central subgroup $A$. Here $H$ is a simply connected compact Lie group. Furthermore $H$ is a product of groups in a list, see [4, p. 346]. If we can find an $S^3$ subgroup of $H$ that intersects $A$ trivially we are done. There are two cases. First assume $H = (S^3)^k$. Then $A$ can not contain the center of $(S^3)^k$ since if it does $G$ will be isomorphic to $SO(3)^k \times T^l$. This being the case it is easy to find a subgroup isomorphic to $S^3$ not intersecting $A$. We do this by checking the list. We have $S^3 = SU(2) \subset SU(n) \quad (n \leq 3)$ intersecting the center trivially since the central element of $SU(n)$ are the diagonal matrices with the same $n$'th root of unity as entry. Similarly $S^3 = SP(1) \subset SP(n) \quad (n \geq 2)$ and $S^3 = SU(2) \subset SO(4) \subset SO(n), \quad (n \geq 5)$ do not contain $-I$ which is the only central element $\not= I$. Furthermore $E_8 \supset E_7 \supset E_6 \supset SU(6) \supset S^3$ and since the center of $E_6$ has order 3, $S^3$ must intersect it trivially and $E_7$ must intersect the center of $E_7$ (cyclic of order 2) trivially. Finally $F_4 \supset SP(3)$ and $G_2 \supset SU(2)$ and these groups have trivial center. We are done.

5.2. Remark. It would be nice to have a conceptual proof of Proposition 5.1. Working in the Lie algebra it is not hard to find a subgroup isomorphic to $SO(3)$ or $S^3$ but it is crucial for us to be in the latter case.

5.3. Proposition. Let $G$ be a compact Lie group with $S^3$ as a subgroup $G \supset S^3$ Then

$$
S^1 \hookrightarrow S^3 \twoheadrightarrow S^2
$$

$$
S^1 \hookrightarrow G \twoheadrightarrow G/S^1
$$

$$
* \hookrightarrow G/S^3 \twoheadrightarrow G/S^3
$$

is a special 1-torus in $G$.

Proof. Lemma 3.4 of [8] shows that $G/S^1$ is stably parallellizable. It follows from [8, lemma 3.3] that $H^3(G;\mathbb{Q}) \to H^3(S^3;\mathbb{Q})$ is onto. Let $G_0 \to K(\mathbb{Q},3) = S^3_0$ represent an element in $H^3(G;\mathbb{Q})$ hitting the generator of $H^3(S^3;\mathbb{Q})$ then one sees by a spectral sequence argument...
that $G_0 \to (G/S^3)_0 \times S^3_0$ is a homology equivalence hence a homotopy equivalence and we are done.

Final Remarks. In case $D(\mathbb{Z}\pi) = 0$ we could replace the concept special 1-torus by the concept 1-torus (see [8]). Since admitting a 1-torus is a weaker condition than admitting a special 1-torus, it is not entirely a loss to have both concepts.

References


Matematisk Institut, Odense Universitet, Danmark