TOPOLOGICAL EQUIVALENCE OF LINEAR REPRESENTATIONS FOR CYCLIC GROUPS: I

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ABSTRACT. In the two parts of this paper we prove that the Reidemeister torsion invariants determine topological equivalence of G–representations, for G a finite cyclic group.

1. Introduction

Let G be a finite group and V, V' finite dimensional real orthogonal representations of G. Then V is said to be *topologically equivalent* to V' (denoted $V \sim_t V'$) if there exists a homeomorphism $h\colon V \to V'$ which is G-equivariant. If V, V' are topologically equivalent, but not linearly isomorphic, then such a homeomorphism is called a non-linear similarity. These notions were introduced and studied by de Rham [30], [31], and developed extensively in [3], [4], [21], [22], and [8]. In the two parts of this paper, referred to as [**I**] and [**II**], we complete de Rham's program by showing that Reidemeister torsion invariants and number theory determine non-linear similarity for finite cyclic groups.

A G-representation is called *free* if each element $1 \neq g \in G$ fixes only the zero vector. Every representation of a finite cyclic group has a unique maximal free subrepresentation.

Theorem. *Let G be a finite cyclic group and V_1, V_2 be free G–representations. For any G–representation W, the existence of a non-linear similarity $V_1 \oplus W \sim_t V_2 \oplus W$ is entirely determined by explicit congruences in the weights of the free summands V_1, V_2, and the ratio $\Delta(V_1)/\Delta(V_2)$ of their Reidemeister torsions, up to an algebraically described indeterminacy.*

The notation and the indeterminacy are given in Section 2 and a detailed statement of results in Theorems A–E. For cyclic groups of 2-power order, we obtain a complete classification of non-linear similarities (see Section 11).

In [3], Cappell and Shaneson showed that non-linear similarities $V \sim_t V'$ exist for cyclic groups $G = C(4q)$ of every order $4q \geqq 8$. On the other hand, if $G = C(q)$ or $G = C(2q)$, for q odd, Hsiang–Pardon [21] and Madsen–Rothenberg [22] proved that topological equivalence of G–representations implies linear equivalence (the case $G = C(4)$ is trivial). Since linear G-equivalence for general finite groups G is detected by restriction to cyclic subgroups, it is reasonable to study this case first. For the rest of the paper, unless otherwise mentioned, G denotes a finite cyclic group.

Further positive results can be obtained by imposing assumptions on the isotropy subgroups allowed in V and V'. For example, de Rham [30] proved in 1935 that piecewise linear similarity implies linear equivalence for free G-representations, by using Reidemeister

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torsion and the Franz Independence Lemma. Topological invariance of Whitehead torsion shows that his method also rules out non–linear similarity in this case. In [17, Thm.A] we studied "first–time" similarities, where $\operatorname{Res}_K V \cong \operatorname{Res}_K V'$ for all proper subgroups $K \subsetneqq G$, and showed that topological equivalence implies linear equivalence if V, V' have no isotropy subgroup of index 2. This result is an application of bounded surgery theory (see [16], [17, §4]), and provides a more conceptual proof of the Odd Order Theorem. These techniques are extended here to provide a necessary and sufficient condition for non-linear similarity in terms of the vanishing of a bounded transfer map (see Theorem 3.5). This gives a new approach to de Rham's problem. The main work of the present paper is to establish methods for effective calculation of the bounded transfer in the presence of isotropy groups of arbitrary index.

An interesting question in non–linear similarity concerns the minimum possible dimension for examples. It is easy to see that the existence of a non–linear similarity $V \sim_t V'$ implies $\dim V = \dim V' \geqq 5$. Cappell, Shaneson, Steinberger and West [8], proved that 6-dimensional similarities exist for $G = C(2^r)$, $r \geqq 4$ and referred to the 1981 Cappell-Shaneson preprint (now published [6]) for the complete proof that 5–dimensional similarities do not exist for any finite group. See Corollary 9.3 for a direct argument using the criterion of Theorem A in the special case of cyclic 2-groups.

In [4], Cappell and Shaneson initiated the study of *stable* topological equvalence for G–representations. We say that V_1 and V_2 are stably topologically similar ($V_1 \approx_t V_2$) if there exists a G–representation W such that $V_1 \oplus W \sim_t V_2 \oplus W$. Let $R_{\mathrm{Top}}(G) = R(G)/R_t(G)$ denote the quotient group of the real representation ring of G by the subgroup $R_t(G) = \{[V_1] - [V_2] \mid V_1 \approx_t V_2\}$. In [4], $R_{\mathrm{Top}}(G) \otimes \mathbf{Z}[1/2]$ was computed, and the torsion subgroup was shown to be 2–primary. As an application of our general results, we determine the structure of the torsion in $R_{\mathrm{Top}}(G)$, for G any cyclic group (see [**II**], Section 13). In Theorem E we give the calculation of $R_{\mathrm{Top}}(G)$ for $G = C(2^r)$. This is the first complete calculation of $R_{\mathrm{Top}}(G)$ for any group that admits non–linear similarities.

Contents

2. Statement of Results

We first introduce some notation, and then give the main results. Let $G = C(4q)$, where $q > 1$, and let $H = C(2q)$ denote the subgroup of index 2 in G. The maximal odd order subgroup of G is denoted G_{odd}. We fix a generator $G = \langle t \rangle$ and a primitive $4q^{th}$-root of unity $\zeta = \exp 2\pi i/4q$. The group G has both a trivial 1-dimensional real representation, denoted $\mathbf{R}_+$, and a non-trivial 1-dimensional real representation, denoted $\mathbf{R}_-$.

A *free* G–representation is a sum of faithful 1-dimensional complex representations. Let t^a, $a \in \mathbf{Z}$, denote the complex numbers $\mathbf{C}$ with action $t \cdot z = \zeta^a z$ for all $z \in \mathbf{C}$. This representation is free if and only if $(a, 4q) = 1$, and the coefficient a is well–defined only modulo $4q$. Since $t^a \cong t^{-a}$ as real G–representations, we can always choose the weights $a \equiv 1 \bmod 4$. This will be assumed unless otherwise mentioned.

Now suppose that $V_1 = t^{a_1} + \cdots + t^{a_k}$ is a free G–representation. The Reidemeister torsion invariant of V_1 is defined as

$$\Delta(V_1) = \prod_{i=1}^{k} (t^{a_i} - 1) \in \mathbf{Z}[t]/\{\pm t^m\} \ .$$

Let $V_2 = t^{b_1} + \cdots + t^{b_k}$ be another free representation, such that $S(V_1)$ and $S(V_2)$ are G-homotopy equivalent. This just means that the products of the weights $\prod a_i \equiv \prod b_i \bmod 4q$. Then the Whitehead torsion of any G-homotopy equivalence is determined by the element

$$\Delta(V_1)/\Delta(V_2) = \frac{\prod (t^{a_i} - 1)}{\prod (t^{b_i} - 1)}$$

since $\mathrm{Wh}(\mathbf{Z}G) \to \mathrm{Wh}(\mathbf{Q}G)$ is monic [25, p.14]. When there exists a G-homotopy equivalence $f\colon S(V_2) \to S(V_1)$ which is freely G-normally cobordant to the identity map on $S(V_1)$, we say that $S(V_1)$ and $S(V_2)$ are freely *G-normally cobordant*. More generally, we say that $S(V_1)$ and $S(V_2)$ are *s–normally cobordant* if $S(V_1 \oplus U)$ and $S(V_2 \oplus U)$ are freely G-normally cobordant for all free G–representations U. This is a necessary condition for non-linear similarity, which can be decided by explicit congruences in the weights (see [34, Thm. 1.2] and [**II**], Section 12).

This quantity, $\Delta(V_1)/\Delta(V_2)$ is the basic invariant determining non-linear similarity. It represents a unit in the group ring $\mathbf{Z}G$, explicitly described for $G = C(2^r)$ by Cappell and Shaneson in [5, §1] using a pull-back square of rings. To state concrete results we need to evaluate this invariant modulo suitable indeterminacy.

The involution $t \mapsto t^{-1}$ induces the identity on $\mathrm{Wh}(\mathbf{Z}G)$, so we get an element

$$\{\Delta(V_1)/\Delta(V_2)\} \in H^0(\mathrm{Wh}(\mathbf{Z}G))$$

where we use $H^i(A)$ to denote the Tate cohomology $H^i(\mathbf{Z}/2; A)$ of $\mathbf{Z}/2$ with coefficients in A.

Let $\mathrm{Wh}(\mathbf{Z}G^-)$ denote the Whitehead group $\mathrm{Wh}(\mathbf{Z}G)$ together with the involution induced by $t \mapsto -t^{-1}$. Then for $\tau(t) = \frac{\prod (t^{a_i}-1)}{\prod (t^{b_i}-1)}$, we compute

$$\tau(t)\tau(-t) = \frac{\prod (t^{a_i} - 1) \prod ((-t)^{a_i} - 1)}{\prod (t^{b_i} - 1) \prod ((-t)^{b_i} - 1)} = \prod \frac{(t^2)^{a_i} - 1}{((t^2)^{b_i} - 1)}$$

which is clearly induced from $\mathrm{Wh}(\mathbf{Z}H)$. Hence we also get a well defined element

$$\{\Delta(V_1)/\Delta(V_2)\} \in H^1(\mathrm{Wh}(\mathbf{Z}G^-)/\mathrm{Wh}(\mathbf{Z}H)) .$$

This calculation takes place over the ring $\Lambda_{2q} = \mathbf{Z}[t]/(1+t^2+\cdots+t^{4q-2})$, but the result holds over $\mathbf{Z}G$ via the involution–invariant pull-back square

$$\begin{array}{ccc} \mathbf{Z}G & \to & \Lambda_{2q} \\ \downarrow & & \downarrow \\ \mathbf{Z}[\mathbf{Z}/2] & \to & \mathbf{Z}/2q[\mathbf{Z}/2] \end{array}$$

Consider the exact sequence of modules with involution:

$$K_1(\mathbf{Z}H) \to K_1(\mathbf{Z}G) \to K_1(\mathbf{Z}H \to \mathbf{Z}G) \to \widetilde{K}_0(\mathbf{Z}H) \to \widetilde{K}_0(\mathbf{Z}G) \tag{2.1}$$

and define $\mathrm{Wh}(\mathbf{Z}H \to \mathbf{Z}G) = K_1(\mathbf{Z}H \to \mathbf{Z}G)/\{\pm G\}$. We then have a short exact sequence

$$0 \to \mathrm{Wh}(\mathbf{Z}G)/\mathrm{Wh}(\mathbf{Z}H) \to \mathrm{Wh}(\mathbf{Z}H \to \mathbf{Z}G) \to \mathbf{k} \to 0$$

where $\mathbf{k} = \ker(\widetilde{K}_0(\mathbf{Z}H) \to \widetilde{K}_0(\mathbf{Z}G))$. Such an exact sequence of $\mathbf{Z}/2$-modules induces a long exact sequence in Tate cohomology. In particular, we have a coboundary map

$$\delta\colon H^0(\mathbf{k}) \to H^1(\mathrm{Wh}(\mathbf{Z}G^-)/\mathrm{Wh}(\mathbf{Z}H)) .$$

Our first result deals with isotropy groups of index 2, as is the case for all the non–linear similarities constructed in [3].

Theorem A. *Let $V_1 = t^{a_1} + \cdots + t^{a_k}$ and $V_2 = t^{b_1} + \cdots + t^{b_k}$ be free G–representations, with $a_i \equiv b_i \equiv 1 \bmod 4$. There exists a topological similarity $V_1 \oplus \mathbf{R}_- \sim_t V_2 \oplus \mathbf{R}_-$ if and only if*

(i) $\prod a_i \equiv \prod b_i \bmod 4q$,
(ii) $\mathrm{Res}_H V_1 \cong \mathrm{Res}_H V_2$, *and*
(iii) *the element $\{\Delta(V_1)/\Delta(V_2)\} \in H^1(\mathrm{Wh}(\mathbf{Z}G^-)/\mathrm{Wh}(\mathbf{Z}H))$ is in the image of the coboundary $\delta\colon H^0(\mathbf{k}) \to H^1(\mathrm{Wh}(\mathbf{Z}G^-)/\mathrm{Wh}(\mathbf{Z}H))$.*

Remark 2.2. The condition (iii) simplifies for G a cyclic 2-group since $H^0(\mathbf{k}) = 0$ in that case (see Lemma 9.1). Theorem A should be compared with [3, Cor.1], where more explicit conditions are given for "first-time" similarities of this kind under the assumption that q is odd, or a 2-power, or $4q$ is a "tempered" number. See also [**II**], Theorem 9.2 for a more general result concerning similarities without $\mathbf{R}_+$ summands. The case $\dim V_1 = \dim V_2 = 4$ gives a reduction to number theory for the existence of 5-dimensional similarities (see Remark 7.2).

Our next result uses a more elaborate setting for the invariant. Let

$$\Phi = \begin{pmatrix} \mathbf{Z}H & \to & \widehat{\mathbf{Z}}_2 H \\ \downarrow & & \downarrow \\ \mathbf{Z}G & \to & \widehat{\mathbf{Z}}_2 G \end{pmatrix}$$

and consider the exact sequence

$$0 \to K_1(\mathbf{Z}H \to \mathbf{Z}G) \to K_1(\widehat{\mathbf{Z}}_2 H \to \widehat{\mathbf{Z}}_2 G) \to K_1(\Phi) \to \widetilde{K}_0(\mathbf{Z}H \to \mathbf{Z}G) \to 0 . \tag{2.3}$$

Again we can define the Whitehead group versions by dividing out trivial units $\{\pm G\}$, and get a double coboundary

$$\delta^2\colon H^1(\widetilde{K}_0(\mathbf{Z}H \to \mathbf{Z}G^-)) \to H^1(\mathrm{Wh}(\mathbf{Z}H \to \mathbf{Z}G^-)) \ .$$

There is a natural map $H^1(\mathrm{Wh}(\mathbf{Z}G^-)/\,\mathrm{Wh}(\mathbf{Z}H)) \to H^1(\mathrm{Wh}(\mathbf{Z}H \to \mathbf{Z}G^-))$, and we will use the same notation $\{\Delta(V_1)/\Delta(V_2)\}$ for the image of the Reidemeister torsion invariant in this new domain. The non-linear similarities handled by the next result have isotropy of index $\leqq 2$.

Theorem B. *Let $V_1 = t^{a_1} + \cdots + t^{a_k}$ and $V_2 = t^{b_1} + \cdots + t^{b_k}$ be free G–representations. There exists a topological similarity $V_1 \oplus \mathbf{R}_- \oplus \mathbf{R}_+ \sim_t V_2 \oplus \mathbf{R}_- \oplus \mathbf{R}_+$ if and only if*

(i) $\prod a_i \equiv \prod b_i \bmod 4q$,
(ii) $\mathrm{Res}_H V_1 \cong \mathrm{Res}_H V_2$, *and*
(iii) *the element $\{\Delta(V_1)/\Delta(V_2)\}$ is in the image of the double coboundary*

$$\delta^2\colon H^1(\widetilde{K}_0(\mathbf{Z}H \to \mathbf{Z}G^-)) \to H^1(\mathrm{Wh}(\mathbf{Z}H \to \mathbf{Z}G^-)) \ .$$

This result can be applied to 6-dimensional similarities.

Corollary 2.4. *Let $G = C(4q)$, with q odd, and suppose that the fields $\mathbf{Q}(\zeta_d)$ have odd class number for all $d \mid 4q$. Then G has no 6-dimensional non-linear similarities.*

Remark 2.5. For example, the class number condition is satisfied for $q \leqq 11$, but not for $q = 29$. The proof is given in [**II**], Section 11. This result corrects [8, Thm.1(i)], and shows that the computations of $R_{\mathrm{Top}}(G)$ given in [8, Thm. 2] are incorrect. We explain the the source of these mistakes in Remark 6.4.

Our final example of the computation of bounded transfers is suitable for determining stable non–linear similarities inductively, with only a minor assumption on the isotropy subgroups. To state the algebraic conditions, we must again generalize the indeterminacy for the Reidemeister torsion invariant to include bounded K–groups (see [**II**], Section 5). In this setting $\widetilde{K}_0(\mathbf{Z}H \to \mathbf{Z}G) = \widetilde{K}_0(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z}))$ and $\mathrm{Wh}(\mathbf{Z}H \to \mathbf{Z}G) = \mathrm{Wh}(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z}))$. We consider the analogous double coboundary

$$\delta^2\colon H^1(\widetilde{K}_0(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))) \to H^1(\mathrm{Wh}(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z})))$$

and note that there is a map $\mathrm{Wh}(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z}) \to \mathrm{Wh}(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))$ induced by the inclusion on the control spaces. We will use the same notation $\{\Delta(V_1)/\Delta(V_2)\}$ for the image of our Reidemeister torsion invariant in this new domain.

Theorem C. *Let $V_1 = t^{a_1} + \cdots + t^{a_k}$ and $V_2 = t^{b_1} + \cdots + t^{b_k}$ be free G–representations. Let W be a complex G–representation with no $\mathbf{R}_+$ summands. Then there exists a topological similarity $V_1 \oplus W \oplus \mathbf{R}_- \oplus \mathbf{R}_+ \sim_t V_2 \oplus W \oplus \mathbf{R}_- \oplus \mathbf{R}_+$ if and only if*

(i) *$S(V_1)$ is s–normally cobordant to $S(V_2)$,*
(ii) $\mathrm{Res}_H(V_1 \oplus W) \oplus \mathbf{R}_+ \sim_t \mathrm{Res}_H(V_2 \oplus W) \oplus \mathbf{R}_+$, *and*
(iii) *the element $\{\Delta(V_1)/\Delta(V_2)\}$ is in the image of the double coboundary*

$$\delta^2\colon H^1(\widetilde{K}_0(\mathcal{C}_{W_{max}\times\mathbf{R}_-,G}(\mathbf{Z}))) \to H^1(\mathrm{Wh}(\mathcal{C}_{W_{max}\times\mathbf{R}_-,G}(\mathbf{Z}))) \ ,$$

where $0 \subseteq W_{max} \subseteq W$ is a complex subrepresentation of real dimension $\leqq 2$, with maximal isotropy group among the isotropy groups of W with 2-power index.

Remark 2.6. The existence of a similarity implies that $S(V_1)$ and $S(V_2)$ are s-normally cobordant. In particular, $S(V_1)$ must be freely G-normally cobordant to $S(V_2)$ and this unstable normal invariant condition is enough to give us a surgery problem. The computation of the bounded transfer in L–theory leads to condition (iii), and an expression of the obstruction to the existence of a similarity purely in terms of bounded K-theory. To carry out this computation we may need to stabilize in the free part, and this uses the s-normal cobordism condition.

Remark 2.7. Theorem C is proved in [**II**], Section 9. Note that $W_{max} = 0$ in condition (iii) if W has no isotropy subgroups of 2-power index. Theorem C suffices to handle stable topological similarities, but leaves out cases where W has an odd number of $\mathbf{R}_-$ summands (handled in [**II**], Theorem 9.2 and the results of [**II**], Section 10). Simpler conditions can be given when $G = C(2^r)$ (see Section 9 in this part).

The double coboundary in (iii) can also be expressed in more "classical" terms by using the short exact sequence

$$0 \to \mathrm{Wh}(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})) \to \mathrm{Wh}(\mathcal{C}_{W_{max}\times\mathbf{R}_-,G}(\mathbf{Z})) \to K_1(\mathcal{C}^{>\mathbf{R}_-}_{W_{max}\times\mathbf{R}_-,G}(\mathbf{Z})) \to 0 \tag{2.8}$$

derived in [**II**], Corollary 6.9. We have $K_1(\mathcal{C}^{>\mathbf{R}_-}_{W_{max}\times\mathbf{R}_-,G}(\mathbf{Z})) = K_{-1}(\mathbf{Z}K)$, where K is the isotropy group of W_{max}, and $\mathrm{Wh}(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})) = \mathrm{Wh}(\mathbf{Z}H \to \mathbf{Z}G)$. The indeterminacy in Theorem C is then generated by the double coboundary

$$\delta^2\colon H^1(\widetilde{K}_0(\mathbf{Z}H \to \mathbf{Z}G^-)) \to H^1(\mathrm{Wh}(\mathbf{Z}H \to \mathbf{Z}G^-))$$

used in Theorem B and the coboundary

$$\delta\colon H^0(K_{-1}(\mathbf{Z}K)) \to H^1(\mathrm{Wh}(\mathbf{Z}H \to \mathbf{Z}G^-))$$

from the Tate cohomology sequence of (2.8).

Finally, we will apply these results to $R_{\mathrm{Top}}(G)$. In Part **II**, Section 3, we will define a subgroup filtration

$$R_t(G) \subseteq R_n(G) \subseteq R_h(G) \subseteq R(G) \tag{2.9}$$

on the real representation ring $R(G)$, inducing a filtration on

$$R_{\mathrm{Top}}(G) = R(G)/R_t(G)\ .$$

Here $R_h(G)$ consists of those virtual elements with no homotopy obstruction to similarity, and $R_n(G)$ the virtual elements with no normal invariant obstruction to similarity (see [**II**], Section 3 for more precise definitions). Note that $R(G)$ has the nice basis $\{t^i, \delta, \epsilon \mid 1 \leqq i \leqq 2q-1\}$, where $\delta = [\mathbf{R}_-]$ and $\epsilon = [\mathbf{R}_+]$.

Let $R^{free}(G) = \{t^a \mid (a, 4q) = 1\} \subset R(G)$ be the subgroup generated by the free representations. To complete the definition, we let $R^{free}(C(2)) = \{\mathbf{R}_-\}$ and $R^{free}(e) = \{\mathbf{R}_+\}$. Then

$$R(G) = \bigoplus_{K \subseteq G} R^{free}(G/K)$$

and this direct sum splitting intersected with the filtration above gives the subgroups $R^{free}_h(G)$, $R^{free}_n(G)$ and $R^{free}_t(G)$. In addition, we can divide out $R^{free}_t(G)$ and obtain subgroups $R^{free}_{h,\mathrm{Top}}(G)$ and $R^{free}_{n,\mathrm{Top}}(G)$ of $R^{free}_{\mathrm{Top}}(G) = R^{free}(G)/R^{free}_t(G)$. By induction on the order of G, we see that it suffices to study the summand $R^{free}_{\mathrm{Top}}(G)$.

Let $\widetilde{R}^{free}(G) = \ker(\text{Res}\colon R^{free}(G) \to R^{free}(G_{odd}))$, and then project into $R_{\text{Top}}(G)$ to define

$$\widetilde{R}^{free}_{\text{Top}}(G) = \widetilde{R}^{free}(G)/R^{free}_t(G) \ .$$

In Part **II**, Section 4 we prove that $\widetilde{R}^{free}_{\text{Top}}(G)$ is *precisely* the torsion subgroup of $R^{free}_{\text{Top}}(G)$, and in [**II**], Section 13 we show that the subquotient $\widetilde{R}^{free}_{n,\text{Top}}(G) = \widetilde{R}^{free}_n(G)/R^{free}_t(G)$ always has exponent two.

Here is a specific computation (correcting [8, Thm. 2]), proved in [**II**], Section 13.

Theorem D. *Let $G = C(4q)$, with $q > 1$ odd, and suppose that the fields $\mathbf{Q}(\zeta_d)$ have odd class number for all $d \mid 4q$. Then $\widetilde{R}^{free}_{\text{Top}}(G) = \mathbf{Z}/4$ generated by $(t - t^{1+2q})$.*

For any cyclic group G, both $R^{free}(G)/R^{free}_h(G)$ and $R^{free}_h(G)/R^{free}_n(G)$ are torsion groups which can be explicitly determined by congruences in the weights (see [**II**], Section 12 and [34, Thm.1.2]).

We conclude this list of sample results with a calculation of $R_{\text{Top}}(G)$ for cyclic 2-groups.

Theorem E. *Let $G = C(2^r)$, with $r \geqq 4$. Then*

$$\widetilde{R}^{free}_{\text{Top}}(G) = \big\langle \alpha_1, \alpha_2, \dots, \alpha_{r-2}, \beta_1, \beta_2, \dots, \beta_{r-3} \big\rangle$$

subject to the relations $2^s\alpha_s = 0$ for $1 \leqq s \leqq r-2$, and $2^{s-1}(\alpha_s + \beta_s) = 0$ for $2 \leqq s \leqq r-3$, together with $2(\alpha_1 + \beta_1) = 0$.

The generators for $r \geqq 4$ are given by the elements

$$\alpha_s = t - t^{5^{2^{r-s-2}}} \quad \text{and} \quad \beta_s = t^5 - t^{5^{2^{r-s-2}}+1} .$$

We remark that $\widetilde{R}^{free}_{\text{Top}}(C(8)) = \mathbf{Z}/4$ generated by $t - t^5$. In Theorem 11.6 we use this information to give a complete topological classification of linear representations for cyclic 2-groups.

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3. A criterion for non–linear similarity

Our approach to the non-linear similarity problem is through bounded surgery theory (see [11][16], [17]). First an elementary observation about topological equivalences for cyclic groups.

Lemma 3.1. *If $V_1 \oplus W \sim_t V_2 \oplus W'$, where V_1, V_2 are free G–representations, and W and W' have no free summands, then there is a G-homeomorphism $h\colon V_1 \oplus W \to V_2 \oplus W$ such that*

$$h| \bigcup_{1 \neq H \leq G} W^H$$

is the identity.

Proof. Let h be the homeomorphism given by $V_1 \oplus W \sim_t V_2 \oplus W'$. We will successively change h, stratum by stratum. For every subgroup K of G, consider the homeomorphism of K-fixed sets

$$f^K : W^K \to W'^K.$$

This is a homeomorphism of G/K, hence of G–representations. As G–representations we can split

$$V_2 \oplus W' = U \oplus W'^K \sim_t U \oplus W^K = V_2 \oplus W''$$

where the similarity uses the product of the identity and $(f^K)^{-1}$. Notice that the composition of f with this similarity is the *identity* on the K-fixed set. Rename W'' as W' and repeat this successively for all subgroups. We end up with $W = W'$ and a G–homeomorphism inducing the identity on the singular set. □

One consequence is

Lemma 3.2. *If $V_1 \oplus W \sim_t V_2 \oplus W$, then there exists a G-homotopy equivalence $S(V_2) \to S(V_1)$.*

Proof. We may assume that W contains no free summand, since a G-homotopy equivalence $S(V_2 \oplus U) \to S(V_1 \oplus U)$, with U a free G-representation, is G-homotopic to $f \times 1$, where $f : S(V_2) \to S(V_1)$ is a G-homotopy equivalence. If we 1-point compactify h, we obtain a G-homeomorphism

$$h^+ : S(V_1 \oplus W \oplus \mathbf{R}) \to S(V_2 \oplus W \oplus \mathbf{R}).$$

After an isotopy, the image of the free G-sphere $S(V_1)$ may be assumed to lie in the complement $S(V_2 \oplus W \oplus \mathbf{R}) - S(W \oplus \mathbf{R})$ of $S(W \oplus \mathbf{R})$ which is G-homotopy equivalent to $S(V_2)$. □

Any homotopy equivalence $f : S(V_2)/G \to S(V_1)/G$ defines an element $[f]$ in the structure set $\mathcal{S}^h(S(V_1)/G)$. We may assume that $\dim V_i \geqq 4$. This element must be non-trivial: otherwise $S(V_2)/G$ would be topologically h-cobordant to $S(V_1)/G$, and Stallings infinite repetition of h-cobordisms trick would produce a homeomorphism $V_1 \to V_2$ contradicting [1, 7.27] (see also [23, 12.12]), since V_1 and V_2 are free representations. More precisely, we use Wall's extension of the Atiyah–Singer equivariant index formula to the topological locally linear case [33]. If $\dim V_i = 4$, we can cross with $\mathbf{CP}^2$ to avoid low–dimensional difficulties. Crossing with W and parameterising by projection on W defines a map from the classical surgery sequence to the bounded surgery exact sequence

(3.3)
$$\begin{array}{ccccc} L^h_n(\mathbf{Z}G) & \longrightarrow & \mathcal{S}^h(S(V_1)/G) & \longrightarrow & [S(V_1)/G, \mathrm{F/Top}] \\ \downarrow & & \downarrow & & \downarrow \\ L^h_{n+k}(\mathcal{C}_{W,G}(\mathbf{Z})) & \longrightarrow & \mathcal{S}^h_b\begin{pmatrix} S(V_1)\times W/G \\ \downarrow \\ W/G \end{pmatrix} & \longrightarrow & [S(V_1) \times_G W, \mathrm{F/Top}] \end{array}$$

The L–groups in the upper row are the ordinary surgery obstruction groups for oriented manifolds and surgery up to homotopy equivalence. In the lower row, we have bounded L–groups (see [**II**], Section 5) corresponding to an orthogonal action $\rho_W : G \to O(W)$, with orientation character given by $\det(\rho_W)$. Our main criterion for non–linear similarities is:

Theorem 3.4. *Let V_1 and V_2 be free G–representations with* $\dim V_i \geqq 2$. *Then, there is a topological equivalence $V_1 \oplus W \sim_t V_2 \oplus W$ if and only if there exists a G–homotopy equivalence $f\colon S(V_2) \to S(V_1)$ such that the element $[f] \in \mathcal{S}^h(S(V_1)/G)$ is in the kernel of the bounded transfer map*

$$trf_W\colon\ \mathcal{S}^h(S(V_1)/G) \to \mathcal{S}^h_b\begin{pmatrix} S(V_1)\times_G W \\ \downarrow \\ W/G \end{pmatrix}.$$

Proof. For necessity, we refer the reader to [17] where this is proved using a version of equivariant engulfing. For sufficiency, we notice that crossing with $\mathbf{R}$ gives an isomorphism of the bounded surgery exact sequences parameterized by W to simple bounded surgery exact sequence parameterized by $W \times \mathbf{R}$. By the bounded s-cobordism theorem, this means that the vanishing of the bounded transfer implies that

$$\begin{array}{ccc} S(V_2)\times W\times \mathbf{R} & \xrightarrow{f\times 1} & S(V_1)\times W\times \mathbf{R} \\ & & \downarrow \\ & & W\times \mathbf{R} \end{array}$$

is within a bounded distance of an equivariant homeomorphism h, where distances are measured in $W \times \mathbf{R}$. We can obviously complete $f \times 1$ to the map

$$f * 1\colon S(V_2) * S(W \times \mathbf{R}) \to S(V_1) * S(W \times \mathbf{R})$$

and since bounded in $W \times \mathbf{R}$ means small near the subset

$$S(W \times \mathbf{R}) \subset S(V_i) * S(W \times \mathbf{R}) = S(V_i \oplus W \oplus \mathbf{R}),$$

we can complete h by the identity to get a G-homeomorphism

$$S(V_2 \oplus W \oplus \mathbf{R}) \to S(V_1 \oplus W \oplus \mathbf{R})$$

and taking a point out we have a G-homeomorphism $V_2 \oplus W \to V_2 \oplus W$ □

By comparing the ordinary and bounded surgery exact sequences (3.3), and noting that the bounded transfer induces the identity on the normal invariant term, we see that a necessary condition for the existence of any stable similarity $f\colon V_2 \approx_t V_1$ is that $f\colon S(V_2) \to S(V_1)$ has s–normal invariant zero. Assuming this, under the natural map

$$L^h_n(\mathbf{Z}G) \to \mathcal{S}^h(S(V_1)/G),$$

where $n = \dim V_1$, the element $[f]$ is the image of $\sigma(f) \in L^h_n(\mathbf{Z}G)$, obtained as the surgery obstruction (relative to the boundary) of a normal cobordism from f to the identity. The element $\sigma(f)$ is well-defined in $\tilde{L}^h_n(\mathbf{Z}G) = \mathrm{Coker}(L^h_n(\mathbf{Z}) \to L^h_n(\mathbf{Z}G))$. Since the image of the normal invariants

$$[S(V_1)/G \times I, S(V_1)/G \times \partial I, \mathrm{F/Top}] \to L^h_n(\mathbf{Z}G)$$

factors through $L^h_n(\mathbf{Z})$ (see [15, Thm.A, 7.4] for the image of the assembly map), we may apply the criterion of 3.4 to any lift $\sigma(f)$ of $[f]$. This reduces the evaluation of the bounded transfer on structure sets to a bounded L–theory calculation.

Theorem 3.5. *Let V_1 and V_2 be free G–representations with* $\dim V_i \geqq 2$. *Then, there is a topological equivalence $V_1 \oplus W \sim_t V_2 \oplus W$ if and only if there exists a G–homotopy equivalence $f\colon S(V_2) \to S(V_1)$, which is G–normally cobordant to the identity, such that $trf_W(\sigma(f)) = 0$, where $trf_W\colon L^h_n(\mathbf{Z}G) \to L^h_{n+k}(\mathcal{C}_{W,G}(\mathbf{Z}))$ is the bounded transfer.*

The rest of the paper is about the computation of these bounded transfer homomorphisms in L–theory. We will need the following result (proved for K_0 in [17, 6.3]).

Theorem 3.6. *Let W be a G–representation with $W^G = 0$. For all $i \in \mathbf{Z}$, the bounded transfer $trf_W\colon K_i(\mathbf{Z}G) \to K_i(\mathcal{C}_{W,G}(\mathbf{Z}))$ is equal to the cone point inclusion $c_*\colon K_i(\mathbf{Z}G) = K_i(\mathcal{C}_{pt,G}(\mathbf{Z})) \to K_i(\mathcal{C}_{W,G}(\mathbf{Z}))$.*

Proof. Let G be a finite group and V a representation. Crossing with V defines a transfer map in K–theory $K_i(RG) \to K_i(\mathcal{C}_{V,G}(R))$ for all i, where R is any ring with unit [16, p.117]. To show that it is equal to the map $K_i(\mathcal{C}_{0,G}(R)) \to K_i(\mathcal{C}_{V,G}(R))$ induced by the inclusion $0 \subset V$, we need to choose models for K–theory.

For RG we choose the category of finitely generated free RG modules, but we think of it as a category with cofibrations and weak equivalences with weak equivalences isomorphisms and cofibrations split inclusions. For $\mathcal{C}_{V,G}(R)$ we use the category of finite length chain complexes, with weak equivalences chain homotopy equivalences and cofibrations sequences that are split short exact at each level. The K–theory of this category is the same as the K-theory of $\mathcal{C}_{V,G}(R)$. For an argument working in this generality see [9].

Tensoring with the chain complex of (V, G) induces a map of categories with cofibrations and weak equivalences hence a map on K–theory. It is elementary to see this agrees with the geometric definition in low dimensions, since the identification of the K–theory of chain complexes of an additive category with the K–theory of the additive category is an Euler characteristic (see e. g. [9]).

By abuse of notation we denote the category of finite chain complexes in $\mathcal{C}_{V,G}(R)$ simply by $\mathcal{C}_{V,G}(R)$. We need to study various related categories. First there is $\mathcal{C}^{iso}_{V,G}(R)$ where we have replaced the weak equivalences by isomorphisms. Obviously the transfer map, tensoring with the chains of (V, G) factors through this category. Also the transfer factors through the category $\mathcal{D}^{iso}_{V,G}(R)$ with the same objects, and isomorphisms as weak equivalences but the control condition is 0-control instead of bounded control. The category $\mathcal{D}^{iso}_{V,G}(R)$ is the product of the full subcategories on objects with support at 0 and the full subcategory on objects with support on $V - 0$, $\mathcal{D}^{iso}_{0,G}(R) \times \mathcal{D}^{iso}_{V-0,G}(R)$, and the transfer factors through chain complexes concentrated in degree 0 in $\mathcal{D}^{iso}_{0,G}(R)$ crossed with chain complexes in the other factor.

But the subcategory of chain complexes concentrated in degree zero of $\mathcal{D}^{iso}_{0,G}(R)$ is precisely the same as $\mathcal{C}_{0,G}(R)$ and the map to $\mathcal{C}_{V,G}(R)$ is induced by inclusion. So to finish the proof we have to show that the other factor $\mathcal{D}^{iso}_{V-0,G}(R)$ maps to zero. For this we construct an intermediate category $\mathcal{E}^{iso}_{V-0,G}(R)$ with the same objects, but where the morphisms are bounded radially and 0-controlled otherwise (i. e. a nontrivial map between objects at different points is only allowed if the points are on the same radial line, and there is a bound on the distance independent of the points). This category has trivial K-theory since we can make a radial Eilenberg swindle toward infinity. Since the other factor $\mathcal{D}^{iso}_{V-0,G}(R)$

maps through this category, we find that the transfer maps through the corner inclusion as claimed. ☐

Remark 3.7. It is an easy consequence of the filtering arguments based on [16, Thm.3.12] that the bounded L–groups are finitely generated abelian groups with 2–primary torsion subgroups. We will therefore localize all the L–groups by tensoring with $\mathbf{Z}_{(2)}$ (without changing the notation), and this loses no information for computing bounded transfers. One concrete advantage of working with the 2–local L–groups is that we can use the idempotent decomposition [13, §6] and the direct sum splitting $L^h_n(\mathcal{C}_{W,G}(\mathbf{Z})) = \oplus_{d|q} L^h_n(\mathcal{C}_{W,G}(\mathbf{Z}))(d)$. Since the "top component" $L^h_n(\mathcal{C}_{W,G}(\mathbf{Z}))(q)$ is just the kernel of the restriction map to all odd index subgroups of G, the use of components is well-adapted to inductive calculations.

A first application of these techniques was given in [17, 5.1].

Theorem 3.8. *For any G–representation W, let $W = W_1 \oplus W_2$ where W_1 is the direct sum of the irreducible summands of W with isotropy subgroups of 2–power index. If $W^G = 0$, then*

(i) *the inclusion $L^h_n(\mathcal{C}_{W_1,G}(\mathbf{Z}))(q) \to L^h_n(\mathcal{C}_{W,G}(\mathbf{Z}))(q)$ is an isomorphism on the top component,*
(ii) *the bounded transfer*

$$trf_{W_2}\colon L^h_n(\mathcal{C}_{W_1,G}(\mathbf{Z}))(q) \to L^h_n(\mathcal{C}_{W,G}(\mathbf{Z}))(q)$$

is an injection on the top component, and
(iii) $\ker(trf_W) = \ker(trf_{W_1}) \subseteq L^h_n(\mathbf{Z}G)(q)$.

Proof. In [17] we localized at an odd prime $p \nmid |G|$ in order to use the Burnside idempotents for all cyclic subgroups of G. The same proof works for the L–groups localized at 2, to show that trf_{W_2} is injective on the top component. ☐

Lemma 3.9. *For any choice of normal cobordism between f and the identity, the surgery obstruction $\sigma(f)$ is a nonzero element of infinite order in $\tilde{L}^h_n(\mathbf{Z}G)$.*

Proof. See [17, 4.5] ☐

The following result (combined with Theorem 3.5) shows that there are no non-linear similarities between semi-free G–representations, since $L^h_{n+1}(\mathcal{C}_{\mathbf{R},G}(\mathbf{Z})) = L^p_n(\mathbf{Z}G)$ and the natural map $L^h_n(\mathbf{Z}G) \to L^p_n(\mathbf{Z}G)$ may be identified with the bounded transfer $trf_{\mathbf{R}}\colon L^h_n(\mathbf{Z}G) \to L^h_{n+1}(\mathcal{C}_{\mathbf{R},G}(\mathbf{Z}))$ [29, §15].

Corollary 3.10. *Under the natural map $L^h_n(\mathbf{Z}G) \to L^p_n(\mathbf{Z}G)$, the image of $\sigma(f)$ is nonzero.*

Proof. The kernel of the map $L^h_n(\mathbf{Z}G) \to L^p_n(\mathbf{Z}G)$ is the image of $H^n(\widetilde{K}_0(\mathbf{Z}G))$ which is a torsion group. ☐

4. Bounded $\mathbf{R}_-$ transfers

Let G denote a finite group of even order, with a subgroup $H < G$ of index 2. We first describe the connection between the bounded $\mathbf{R}_-$ transfer and the compact line bundle transfer of [33, 12C] by means of the following diagram:

$$\begin{array}{ccc} L^h_n(\mathbf{Z}G, w) & \xrightarrow{trf_{I_-}} & L^h_{n+1}(\mathbf{Z}H \to \mathbf{Z}G, w\phi) \\ \downarrow{\scriptstyle trf_{\mathbf{R}_-}} & & \downarrow{\scriptstyle j_*} \\ L^h_{n+1}(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z}), w\phi) & \xleftarrow[r_*]{} & L^{k,h}_{n+1}(\mathbf{Z}H \to \mathbf{Z}G, w\phi) \end{array}$$

where $w\colon G \to \{\pm 1\}$ is the orientation character for G and $\phi\colon G \to \{\pm 1\}$ has kernel H. On $\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})$ we start with the standard orientation defined in [**II**], Example 5.4, and then twist by w or $w\phi$. Note that the (untwisted) orientation induced on $\mathcal{C}_{pt}(\mathbf{Z}G)$ via the cone point inclusion $c\colon \mathcal{C}_{pt}(\mathbf{Z}G) \to \mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})$ is *non–trivial.* The homomorphism

$$r_*\colon L^{k,h}_{n+1}(\mathbf{Z}H \to \mathbf{Z}G, w\phi) \to L^h_{n+1}(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z}), w\phi)$$

is obtained by adding a ray $[1, \infty)$ to each point of the boundary double cover in domain and range of a surgery problem. Here $\mathbf{k}$ in the decoration means that we are allowing projective $\mathbf{Z}H$-modules that become free when induced up to $\mathbf{Z}G$

Theorem 4.1. *The map $r_*\colon L^{k,h}_{n+1}(\mathbf{Z}H \to \mathbf{Z}G, w\phi) \to L^h_{n+1}(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z}), w\phi)$ is an isomorphism, and under this identification, the bounded $\mathbf{R}_-$ transfer corresponds to the line bundle transfer, followed by the relaxation of projectivity map j_* given by $\mathbf{k}$.*

Proof. Let $\mathcal{A}$ be the full subcategory of $\mathcal{U} = \mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})$ with objects that are only nontrivial in a bounded neighborhood of 0. Then $\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})$ is $\mathcal{A}$-filtered. The category $\mathcal{A}$ is equivalent to the category of free $\mathbf{Z}G$-modules (with the non-orientable involution). The quotient category $\mathcal{U}/\mathcal{A}$ is equivalent to $\mathcal{C}^{>0}_{[0,\infty),H}(\mathbf{Z})$, which has the same L-theory as $\mathcal{C}_{\mathbf{R}}(\mathbf{Z}H)$, so we get a fibration of spectra

$$\mathbb{L}^k(\mathbf{Z}H) \to \mathbb{L}^h(\mathbf{Z}G) \to \mathbb{L}^h(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})) \ .$$

This shows that

$$\mathbb{L}^h(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})) \simeq \mathbb{L}^{k,h}(\mathbf{Z}H \to \mathbf{Z}G)$$

□

The line bundle transfer can be studied by the long exact sequence

$$(4.2)\quad \cdots \to LN_n(\mathbf{Z}H \to \mathbf{Z}G, w\phi) \to L^h_n(\mathbf{Z}G, w) \to L^h_{n+1}(\mathbf{Z}H \to \mathbf{Z}G, w\phi) \\ \to LN_{n-1}(\mathbf{Z}H \to \mathbf{Z}G, w\phi) \to L^h_{n-1}(\mathbf{Z}G, w) \to \dots$$

given in [33, 11.6]. The obstruction groups $LN_n(\mathbf{Z}H \to \mathbf{Z}G, w\phi)$ for codimension 1 surgery have an algebraic description

$$(4.3)\qquad LN_n(\mathbf{Z}H \to \mathbf{Z}G, w\phi) \cong L^h_n(\mathbf{Z}H, \alpha, u)$$

given by [33, 12.9]. The groups on the right–hand side are the algebraic L–groups of the "twisted" anti–structure defined by choosing some element $t \in G - H$ and then setting $\alpha(x) = w(x)t^{-1}x^{-1}t$ for all $x \in H$, and $u = w(t)t^{-2}$. Another choice of $t \in G - H$ gives a scale equivalent anti–structure on $\mathbf{Z}H$. The same formulas also give a "twisted" anti–structure $(\mathbf{Z}G, \alpha, u)$ on $\mathbf{Z}G$, but since the conjugation by t is now an inner automorphism of G this is scale equivalent to the standard structure $(\mathbf{Z}G, w)$. We can therefore define the twisted induction map

$$\tilde{i}_*\colon L^h_n(\mathbf{Z}H, \alpha, u) \to L^h_n(\mathbf{Z}G, w)$$

and the twisted restriction map

$$\tilde{\gamma}_*\colon L^h_n(\mathbf{Z}G, w) \to L^h_n(\mathbf{Z}H, \alpha, u)$$

as the composites of the ordinary induction or restriction maps (induced by the inclusion $(\mathbf{Z}H, w) \to (\mathbf{Z}G, w)$) with the scale isomorphism.

The twisted anti–structure on $\mathbf{Z}H$ is an example of a "geometric anti–structure" [19, p.110]:

$$\alpha(g) = w(g)\theta(g^{-1}), \qquad u = \pm b$$

where $\theta\colon G \to G$ is a group automorphism with $\theta^2(g) = bgb^{-1}$, $w \circ \theta = w$, $w(b) = 1$ and $\theta(b) = b$.

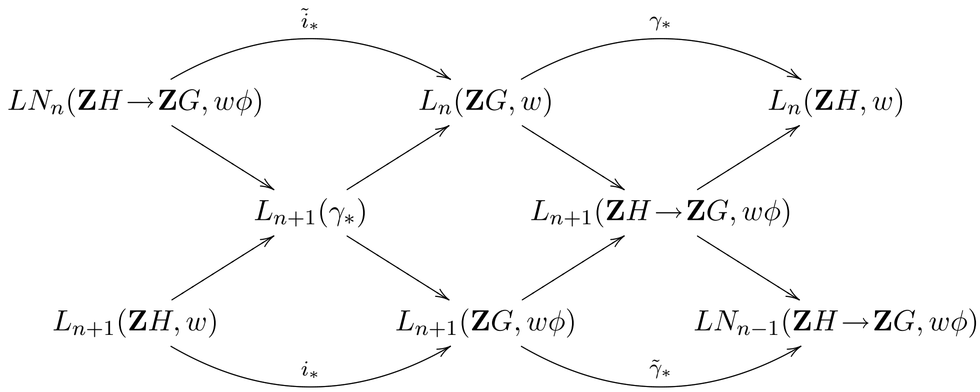
Example 4.4. For G cyclic, the orientation character restricted to H is trivial, $\theta(g) = tgt^{-1} = g$ and $u = w(t)t^2$. Choosing $t \in G$ a generator we get $b = t^2$, which is a generator for H.

There is an identification [12, Thm. 3], [18, 50–53] of the exact sequence (4.2) for the line bundle transfer, extending the scaling isomorphism $L^h_n(\mathbf{Z}G, w) \cong L^h_n(\mathbf{Z}G, \alpha, u)$ and (4.3), with the long exact sequence of the "twisted" inclusion

$$\dots L^h_n(\mathbf{Z}H, \alpha, u) \xrightarrow{\tilde{i}_*} L^h_n(\mathbf{Z}G, \alpha, u) \to L^h_n(\mathbf{Z}H \to \mathbf{Z}G, \alpha, u) \to L^h_{n-1}(\mathbf{Z}H, \alpha, u) \to \dots$$

These identifications can then be substituted into the following "twisting diagram" in order to compute the various maps (see [18, Appendix 2] for a complete tabulation in the case of finite 2–groups).

(4.5)



The existence of the diagram depends on the identifications $L_{n+1}(\gamma_*) \cong L_n(\tilde{\gamma}_*)$ and $L_{n+1}(i_*) \cong L_n(\tilde{i}_*)$ obtained geometrically in [12] and algebraically in [28].

5. Some basic facts in K- and L-theory

In this section we record various calculational facts from the literature about K– and L–theory of cyclic groups. A general reference for the K–theory is [25], and for L–theory computations is [20]. Recall that $\widetilde{K}_0(\mathcal{A}) = K_0(\mathcal{A}^\wedge)/K_0(\mathcal{A})$ for any additive category $\mathcal{A}$, and $\mathrm{Wh}(\mathcal{A})$ is the quotient of $K_1(\mathcal{A})$ by the subgroup defined by the system of stable isomorphisms.

Theorem 5.1. *Let G be a cyclic group, K a subgroup. We then have*

(i) $K_1(\mathbf{Z}G) = (\mathbf{Z}G)^* \subset K_1(\mathbf{Q}G)$ *Here* $(\mathbf{Z}G)^*$ *denotes the units of* $\mathbf{Z}G$.
(ii) *The torsion in* $K_1(\mathbf{Z}G))$ *is precisely* $\{\pm G\}$, *so* $\mathrm{Wh}(\mathbf{Z}G)$ *is torsion free.*
(iii) *The maps* $K_1(\mathbf{Z}K) \to K_1(\mathbf{Z}G)$ *and*
$$\mathrm{Wh}(\mathbf{Z}G)/\,\mathrm{Wh}(\mathbf{Z}K) \to \mathrm{Wh}(\mathbf{Q}G)/\,\mathrm{Wh}(\mathbf{Q}K)$$
are injective.
(iv) $\widetilde{K}_0(\mathbf{Z}G)$ *a torsion group and the map* $\widetilde{K}_0(\mathbf{Z}G) \to \widetilde{K}_0(\mathbf{Z}_{(p)}G$ *is the zero map for all primes p.*
(v) $K_{-1}(\mathbf{Z}G)$ *is torsion free, and sits in an exact sequence*
$$0 \to K_0(\mathbf{Z}) \to K_0(\widehat{\mathbf{Z}}G) \oplus K_0(\mathbf{Q}G) \to K_0(\widehat{\mathbf{Q}}G) \to K_{-1}(\mathbf{Z}G) \to 0\ .$$
(vi) $K_{-1}(\mathbf{Z}K) \to K_{-1}(\mathbf{Z}G)$ *is an injection.*
(vii) $K_{-j}(\mathbf{Z}G) = 0$ *for* $j \geqq 2$.

Proof. The proof mainly consists of references. See [25, pp.6,14] for the first two parts. Part (iii) follows from (i) and the relation $(\mathbf{Z}G)^* \cap (\mathbf{Q}K)^* = (\mathbf{Z}K)^*$. Part (iv) is due to Swan [32], and part (vii) is a result of Bass and Carter [10]. Part (v) gives the arithmetic sequence for computing $K_{-1}(\mathbf{Z}G)$, and the assertion that $K_{-1}(\mathbf{Z}G)$ is torsion free is easy to deduce (see also [10]). Since $\mathrm{Res}_K \circ \mathrm{Ind}_K$ is multiplication by the index $[G : K]$, part (vi) follows from (v). □

Tate cohomology of K_i-groups play an important role. The involution on K–theory is induced by duality on modules. It is conventionally chosen to have the boundary map
$$K_1(\widehat{\mathbf{Q}}(G) \to \tilde{K}_0(\mathbf{Z}G)$$
preserve the involution, so to make this happen we choose to have the involution on K_0 be given by sending $[P]$ to $-[P^*]$, and the involution on K_1 is given by sending τ to τ^*. This causes a shift in dimension in Ranicki-Rothenberg exact sequences
$$\ldots \to H^0(\widetilde{K}_0(\mathcal{A})) \to L^h_{2k}(\mathcal{A}) \to L^p_{2k}(\mathcal{A}) \to H^1(\widetilde{K}_0(\mathcal{A})) \to \ldots$$
compared to
$$\ldots \to H^1(\mathrm{Wh}(\mathcal{A})) \to L^s_{2k}(\mathcal{A}) \to L^h_{2k}(\mathcal{A}) \to H^0(\mathrm{Wh}(\mathcal{A})) \to \ldots$$
and
$$\ldots \to H^1(K_{-1}(\mathcal{A})) \to L^p_{2k}(\mathcal{A}) \to L^{\langle -1\rangle}_{2k}(\mathcal{A}) \to H^0(K_{-1}(\mathcal{A})) \to \ldots$$

Theorem 5.2. *Let G be a cyclic group, K a subgroup.*

(i) $L^s_{2k}(\mathbf{Z}G)$, $L^p_{2k}(\mathbf{Z}G)$, *and* $L^{\langle -1\rangle}_{2k}(\mathbf{Z}G)$ *are torsion–free when k is even, and when k is odd the only torsion is a* $\mathbf{Z}/2$*-summand generated by the Arf invariant element.*

(ii) *The groups* $L^h_{2k+1}(\mathbf{Z}G) = L^s_{2k+1}(\mathbf{Z}G) = L^p_{2k+1}(\mathbf{Z}G)$ *are zero (k even), or* $\mathbf{Z}/2$ *(if k odd and* $|G|$ *is even), detected by projection* $G \to C(2)$.

(iii) $L^{\langle -1\rangle}_{2k+1}(\mathbf{Z}G) = H^1(K_{-1}(\mathbf{Z}G))$ *(k even), or* $\mathbf{Z}/2 \oplus H^1(K_{-1}(\mathbf{Z}G))$ *(k odd).*

(iv) *The Ranicki-Rothenberg exact sequence gives*

$$0 \to H^0(\widetilde{K}_0(\mathbf{Z}G)) \to L^h_{2k}(\mathbf{Z}G) \to L^p_{2k}(\mathbf{Z}G) \to H^1(\widetilde{K}_0(\mathbf{Z}G)) \to 0$$

so $L^h_{2k}(\mathbf{Z}G)$ *has the torsion subgroup* $H^0(\widetilde{K}_0(\mathbf{Z}G))$.

(v) *The double coboundary* $\delta^2\colon H^0(\widetilde{K}_0(\mathbf{Z}G)) \to H^0(\mathrm{Wh}(\mathbf{Z}G))$ *is injective.*

(vi) *The maps* $L^s_{2k}(\mathbf{Z}K) \to L^s_{2k}(\mathbf{Z}G)$, $L^p_{2k}(\mathbf{Z}K) \to L^p_{2k}(\mathbf{Z}G)$ *, and* $L^{\langle -1\rangle}_{2k}(\mathbf{Z}K) \to L^{\langle -1\rangle}_{2k}(\mathbf{Z}G)$ *are injective when k is even or* $[G : K]$ *is odd. For k odd and* $[G : K]$ *even, the kernel is generated by the Arf invariant element.*

(vii) *In the oriented case,* $\mathrm{Wh}(\mathbf{Z}G)$ *has trivial involution and* $H^1(\mathrm{Wh}(\mathbf{Z}G)) = 0$.

Proof. See [20, §3, §12] for the proof of part (i) for L^s or L^p. Part (ii) is due to Bak for L^s and L^h [2], and is proved in [20, 12.1] for L^p. We can now substitute this information into the Ranicki-Rothenberg sequences above to get part (iv). Furthermore, we see that the maps $L^{\langle -1\rangle}_n(\mathbf{Z}G) \to H^n(K_{-1}(\mathbf{Z}G))$ are all surjective, and the extension giving $L^{\langle -1\rangle}_{2k+1}(\mathbf{Z}G)$ actually splits. This gives part (iii). For part (v) we use the fact that the double coboundary $\delta^2\colon H^0(\widetilde{K}_0(\mathbf{Z}G)) \to H^0(\mathrm{Wh}(\mathbf{Z}G))$ can be identified with the composite

$$H^0(\widetilde{K}_0(\mathbf{Z}G)) \to L^h_0(\mathbf{Z}G) \to H^0(\mathrm{Wh}(\mathbf{Z}G))$$

(see [**II**], Section 7). Part (vii) is due to Wall [25].

For $L^{\langle -1\rangle}_{2k}(\mathbf{Z}G)$ we use the exact sequence

$$0 \to L^{\langle -1\rangle}_{2k}(\mathbf{Z}G) \to L^p_{2k}(\widehat{\mathbf{Z}}G) \oplus L^p_{2k}(\mathbf{Q}G) \to L^p_{2k}(\widehat{\mathbf{Q}}G)$$

obtained from the braid of exact sequences given in [13, 3.11] by substituting the calculation $L^p_{2k+1}(\widehat{\mathbf{Q}}G) = 0$ from [14, 1.10]. It is also convenient to use the idempotent decomposition (as in [13, §7]) for $G = C(2^r q)$, q odd:

$$L^{\langle -1\rangle}_{2k}(\mathbf{Z}G) = \bigoplus_{d|q} L^{\langle -1\rangle}_{2k}(\mathbf{Z}G)(d)$$

where the d-component, $d \neq q$, is mapped isomorphically under restriction to $L^p_{2k+1}(\mathbf{Z}K, w)(d)$ for $K = C(2^r d)$. This decomposition extends to a decomposition of the arithmetic sequence above. The summand corresponding to $d = 1$ may be neglected since $L^p = L^{\langle -1\rangle}$ for a 2-group (since the K_{-1} vanishes in that case).

We now study $L^p_{2k}(\mathbf{Q}G)$ by comparing it to $L^p_{2k}(\widehat{\mathbf{Q}}G) \oplus L^p_{2k}(\mathbf{R}G)$ as in [14, 1.13]. Let $CL^p_n(S) = L^p_n(S \to S_A)$, where S is a factor of $\mathbf{Q}G$, and $S_A = \widehat{S} \oplus (S \otimes \mathbf{R})$. If S has type U, we obtain $CL^p_{2k+1}(S) = 0$, and we have an extension $0 \to \mathbf{Z}/2 \to CL^p_{2k}(S) \to H^1(K_0(S_A)/K_0(S)) \to 0$. We may now assume that $q > 1$, implying that all the factors in the q-component of $\mathbf{Q}G$ have type U. By induction on q it is enough to consider the q-component of the exact sequence above. It can be re-written in the form

$$0 \to L^{\langle -1\rangle}_{2k}(\mathbf{Z}G)(q) \to L^p_{2k}(\widehat{\mathbf{Z}}G)(q) \oplus L^p_{2k}(\mathbf{R}G)(q) \to CL^p_{2k}(\mathbf{Q}G)(q)$$

But $L^p_{2k}(\widehat{\mathbf{Z}}G)(q) \cong H^1(K_0(\widehat{\mathbf{Z}}G)(q))$ by [14, 1.11], and the group $H^1(K_0(\widehat{\mathbf{Z}}G)(q))$ injects into $CL^p_{2k}(\mathbf{Q}G)(q)$. To see this we use the exact sequence in Theorem 5.1 (v), and the fact that the involution on $K_0(\mathbf{Q}G)$ is multiplication by -1. We conclude that $L^{\langle -1\rangle}_{2k}(\mathbf{Z}G)(q)$ injects into $L^p_{2k}(\mathbf{R}G)(q)$ which is torsion-free by [14, 1.9]. Part (vi) now follows from part (i) and the property $\mathrm{Res}_K \circ \mathrm{Ind}_K = [G : K]$. □

6. The computation of $L^p_1(\mathbf{Z}G, w)$

Here we correct an error in the statement of [14, 5.1] (Notice however that Table 2 [14, p.553] has the correct answer).

Proposition 6.1. *Let $G = \sigma \times \rho$, where σ is an abelian 2–group and ρ has odd order. Then $L^p_n(\mathbf{Z}G, w) = L^p_n(\mathbf{Z}\sigma, w) \oplus L^p_n(\mathbf{Z}\sigma \to \mathbf{Z}G, w)$ where $w\colon G \to \{\pm 1\}$ is an orientation character. For $i = 2k$, the second summand is free abelian and detected by signatures at the type $U(\mathbf{C})$ representations of G which are non–trivial on ρ. For $n = 2k + 1$, the second summand is a direct sum of $\mathbf{Z}/2$'s, one for each type $U(\mathbf{GL})$ representation of G which is non–trivial on ρ.*

Remark 6.2. Note that type $U(\mathbf{C})$ representations of G exist only when $w \equiv 1$, and type $U(\mathbf{GL})$ representations of G exist only when $w \not\equiv 1$. In both cases, the second summand is computed by transfer to cyclic subquotients of order $2^r q$, $q > 1$ odd, with $r \geqq 2$.

Proof. The given direct sum decomposition follows from the existence of a retraction of the inclusion $\sigma \to G$ compatible with w. It also follows that

$$L^{p,h}_{n+1}(\mathbf{Z}G \to \widehat{\mathbf{Z}}_2 G, w) \cong L^{p,h}_{n+1}(\mathbf{Z}\sigma \to \widehat{\mathbf{Z}}_2\sigma, w) \oplus L^p_n(\mathbf{Z}\sigma \to \mathbf{Z}G, w)$$

since the map $L^h_n(\widehat{\mathbf{Z}}_2\sigma, w) \to L^h_n(\widehat{\mathbf{Z}}_2 G, w)$ is an isomorphism. The computation of the relative groups for $\mathbf{Z} \to \widehat{\mathbf{Z}}_2$ can be read off from [14, Table 2,Remark 2.14]: for each centre field E of a type $U(\mathbf{GL})$ representation, the contribution is $H^0(C(E)) \cong \mathbf{Z}/2$ if $i \equiv 1 \bmod 2$.

The detection of $L^p_i(\mathbf{Z}\sigma \to \mathbf{Z}G, w)$ by cyclic subquotients is proved in [19, 1.B.7, 3.A.6, 3.B.2]. □

Corollary 6.3. *Let $G = C(2^r q)$, for $q > 1$ odd and $r \geqq 2$. Then the group*

$$L^p_{2k+1}(\mathbf{Z}G, w) = \bigoplus_{d|q} L^p_{2k+1}(\mathbf{Z}G, w)(d)$$

where the d-component, $d \neq q$, is mapped isomorphically under restriction to $L^p_{2k+1}(\mathbf{Z}K, w)(d)$ for $K = C(2^r d)$. The q-component is given by the formula

$$L^p_{2k+1}(\mathbf{Z}G, w)(q) = \bigoplus_{i=2}^{r} CL^K_2(E_i) \cong (\mathbf{Z}/2)^{r-1}$$

when $w \not\equiv 1$, where the summand $CL^K_2(E_i) = H^0(C(E_i))$, $2 \leqq i \leqq r$, corresponds to the type $U(\mathbf{GL})$ rational representation with centre field $E_i = \mathbf{Q}(\zeta_{2^i q})$.

Remark 6.4. The calculation of L^p_1 contradicts the assertion in [8, p.733,l.-8] that the projection map $G \to C(2^r)$ induces an isomorphism on L^p_1 in the non–oriented case. In fact, the projection detects only the $q = 1$ component. This error invalidates the proofs of the main results of [8] for cyclic groups *not* of 2–power order, so the reader should not rely on the statements. In particular, we have already noted that [8, Thm. 1(i)] and [8, Thm. 2] are incorrect. On the other hand, the conclusions of [8, Thm. 1] are correct for

6-dimensional similarities of $G = C(2^r)$. We will use [8, Cor.(iii)] in Example 9.8 and in Section 10.

Remark 6.5. The $q = 1$ component, $L^p_{2k+1}((\mathbf{Z}G, w)(1)$, is isomorphic to via the projection or restriction map to $L^p_{2k+1}((\mathbf{Z}[C(2^r)], w)$. In this case, the representation with centre field $\mathbf{Q}(i)$ has type $OK(\mathbf{C})$ and contributes $(\mathbf{Z}/2)^2$ to L^p_3, hence $L^p_1(\mathbf{Z}G, w)(1) \cong (\mathbf{Z}/2)^{r-2}$ and $L^p_3(\mathbf{Z}G, w)(1) \cong (\mathbf{Z}/2)^r$.

We now return to our main calculational device for determining non–linear similarities of cyclic groups, namely the "double coboundary"

$$\delta^2 \colon H^1(\widetilde{K}_0(\mathbf{Z}G^-)) \to H^1(\mathrm{Wh}(\mathbf{Z}G^-))$$

from the exact sequence

$$0 \to \mathrm{Wh}(\mathbf{Z}G) \to \mathrm{Wh}(\widehat{\mathbf{Z}}G) \oplus K_1(\mathbf{Q}G) \to K_1(\widehat{\mathbf{Q}}G) \to \widetilde{K}_0(\mathbf{Z}G) \to 0\ .$$

We recall that the discriminant induces an isomorphism

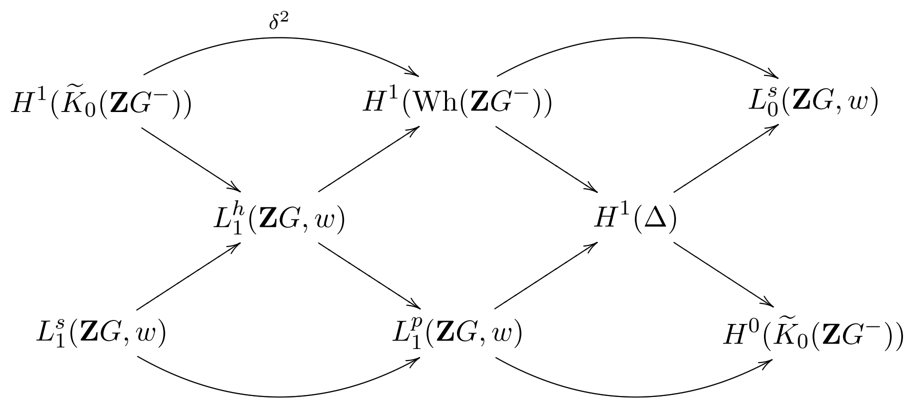
$$L^h_1(\mathbf{Z}G, w) \cong H^1(\mathrm{Wh}(\mathbf{Z}G), w)$$

since $L^s_i(\mathbf{Z}G, w) \cong L'_i(\mathbf{Z}G, w) = 0$ for $i \equiv 0, 1 \mod 4$ by the calculations of [33, 3.4.5,5.4].

Proposition 6.6. *The kernel of the map* $L^h_1(\mathbf{Z}G, w) \to L^p_1(\mathbf{Z}G, w)$ *is isomorphic to the image of the double coboundary* $\delta^2 \colon H^1(\widetilde{K}_0(\mathbf{Z}G^-)) \to H^1(\mathrm{Wh}(\mathbf{Z}G^-))$ *under the isomorphism* $L^h_1(\mathbf{Z}G, w) \cong H^1(\mathrm{Wh}(\mathbf{Z}G^-))$ *induced by the discriminant .*

Proof. We will use the commutative braid

(6.7)



relating the L^h to L^p and the L^s to L^h Rothenberg sequences. The term $H^1(\Delta)$ is the Tate cohomology of the relative group for the double coboundary defined in [**II**], Section 7. The braid diagram is constructed by diagram chasing using the interlocking K and L–theory exact sequences, as in for example [13, §3], [14, p.560], [26, p.3] and [27, 6.2]. We see that the discriminant of an element $\sigma \in L^h_1(\mathbf{Z}G, w)$ lies in the image of the double coboundary if and only if $\sigma \in \ker(L^h_1(\mathbf{Z}G, w) \to L^p_1(\mathbf{Z}G, w))$. ☐

The braid diagram in this proof also gives:

Corollary 6.8. *There is an isomorphism* $L^p_1(\mathbf{Z}G, w) \cong H^1(\Delta)$.

Remark 6.9. It follows from Corollary 6.3 that $H^1(\Delta)$ is fixed by the induced maps from group automorphisms of G. We will generalize this result in the next section.

Remark 6.10. There is a version of these results for $L^p_3(\mathbf{Z}G, w)$ as well, on the kernel of the projection map $L^p_3(\mathbf{Z}G, w) \to L^p_3(\mathbf{Z}K, w)$, where $K = C(4)$. The point is that $L^s_i(\mathbf{Z}G, w) \cong L^s_i(\mathbf{Z}K, w)$ is an isomorphism for $i \equiv 2, 3 \mod 4$ as well [33, 3.4.5,5.4]. There is also a corresponding braid [**II**] (9.1) for $L^s_{2k+1}(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))$, $L^h_{2k+1}(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))$ and $L^p_{2k+1}(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))$ involving the double coboundary in bounded K–theory. The cone point inclusion

$$\mathcal{C}_{pt}(\mathbf{Z}G, w) = \mathcal{C}_{pt,G^-}(\mathbf{Z}) \to \mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z})$$

induces a natural transformation between the two braid diagrams.

In Section 7 we will need the following calculation. We denote by $L_n^{\mathrm{Wh}(\mathbf{Z}H)}(\mathbf{Z}G^-)$ the L–group of $\mathbf{Z}G$ with the non-oriented involution, and Whitehead torsions allowed in the subgroup $\mathrm{Wh}(\mathbf{Z}H) \subset \mathrm{Wh}(\mathbf{Z}G)$.

Lemma 6.11. $L_1^{\mathrm{Wh}(\mathbf{Z}H)}(\mathbf{Z}G^-) = 0$*, and the map* $L_0^{\mathrm{Wh}(\mathbf{Z}H)}(\mathbf{Z}G^-) \to H^0(\mathrm{Wh}(\mathbf{Z}H))$ *induced by the discriminant is an injection.*

Proof. The Rothenberg sequence gives

$$L^s_n(\mathbf{Z}G^-) \to L_n^{\mathrm{Wh}(\mathbf{Z}H)}(\mathbf{Z}G^-) \to H^n(\mathrm{Wh}(\mathbf{Z}H))\ .$$

For $n \equiv 1 \bmod 4$ the outside terms are zero, and hence $L_1^{\mathrm{Wh}(\mathbf{Z}H)}(\mathbf{Z}G^-) = 0$. For $n \equiv 0 \bmod 4$, $L^s_0(\mathbf{Z}G^-) = 0$ as noted above and the injectivity follows. □

In later sections, it will be convenient to stabilize with trivial representations and use the identification

$$L^p_{n+k}(\mathcal{C}_{W\times\mathbf{R}^k,G}(\mathbf{Z})) \cong L_n^{\langle -k\rangle}(\mathcal{C}_{W,G}(\mathbf{Z})).$$

The composite with the transfer

$$trf_{\mathbf{R}^k}\colon L^p_n(\mathbf{Z}G) \to L^p_{n+k}(\mathcal{C}_{W\times\mathbf{R}^k,G}(\mathbf{Z}))$$

is just the usual "change of K–theory" map, which may be analysed by the Ranicki-Rothenberg sequences [29]. For G a finite group, $K_{-j}(\mathbf{Z}G) = 0$ if $j \geqq 2$ so only the first stabilization is needed.

Lemma 6.12. *For* $G = C(2^r q)$ *and* $w\colon G \to \{\pm 1\}$ *non–trivial, the map*

$$L^p_{2k+1}(\mathbf{Z}G, w) \to L^{\langle -1\rangle}_{2k+1}(\mathbf{Z}G, w)$$

is injective.

Proof. The group $K_{-1}(\mathbf{Z}G)$ is a torsion free quotient of $K_0(\widehat{\mathbf{Q}}G)$, and has the involution induced by $[P] \mapsto -[P^*]$ on $K_0(\widehat{\mathbf{Q}}G)$ [13, 3.6]. This implies first that $H^0(K_0(\widehat{\mathbf{Q}}G)) = 0$, and so the image of the coboundary

$$H^0(K_{-1}(\mathbf{Z}G)) \to H^1(K_0(\widehat{\mathbf{Z}}G)) \oplus H^1(K_0(\mathbf{Q}G))$$

consists of the classes $(0, [E])$ where E splits at every finite prime dividing $2q$. We need to compare the exact sequences in the following diagram (see [14], [20]):

$$\begin{array}{ccccccc}
 & & & & L^K_{2k+2}(\widehat{\mathbf{Z}}G, w) \oplus L^K_{2k+2}(\mathbf{Q}G, w) & \longrightarrow & L^K_{2k+2}(\widehat{\mathbf{Q}}G, w) \\
 & & & & \downarrow & & \downarrow \\
0 & \longrightarrow & L^{\langle -1 \rangle}_{2k+2}(\mathbf{Z}G, w) & \longrightarrow & L^p_{2k+2}(\widehat{\mathbf{Z}}G, w) \oplus L^p_{2k+2}(\mathbf{Q}G, w) & \longrightarrow & L^p_{2k+2}(\widehat{\mathbf{Q}}G, w) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & H^0(K_{-1}(\mathbf{Z}G)) & \longrightarrow & H^1(K_0(\widehat{\mathbf{Z}}G)) \oplus H^1(K_0(\mathbf{Q}G)) & \longrightarrow & H^1(K_0(\widehat{\mathbf{Q}}G))
\end{array}$$

The groups $L^K_{2k+2}(\widehat{\mathbf{Z}}G, w)$ reduce to the L–groups of finite fields, which are zero in type U, and the map $L^p_{2k+2}(\mathbf{Q}G, w) \to H^1(K_0(\mathbf{Q}G))$ is surjective. For each involution invariant field E in the top component of $\mathbf{Q}G$, the group $L^K_{2k+2}(E) = H^0(E^\times)$ which maps injectively into $L^K_{2k+2}(\hat{E}) = H^0(\hat{E}^\times)$, [14]. It follows that the images of $L^K_{2k+2}(E)$ and $L^{\langle -1 \rangle}_{2k+2}(\mathbf{Z}G, w)$ in $L^p_{2k+2}(E)$ have zero intersection , and so the composite map $L^{\langle -1 \rangle}_{2k+2}(\mathbf{Z}G, w) \to H^1(K_0(\mathbf{Q}G))$ is an isomorphism onto the classes which split at all primes dividing $2q$. Therefore the map $L^{\langle -1 \rangle}_{2k+2}(\mathbf{Z}G, w) \to H^0(K_{-1}(\mathbf{Z}G))$ is surjective, and we conclude that the map $L^p_{2k+1}(\mathbf{Z}G, w) \to L^{\langle -1 \rangle}_{2k+1}(\mathbf{Z}G, w)$ is injective. □

Corollary 6.13. *Let $G = C(2^r q)$ and $w \colon G \to \{\pm 1\}$ the non–trivial orientation. If W is a G–representation with $W^G = 0$, then the map*

$$L^p_{2k+1}(\mathcal{C}_{W,G}(\mathbf{Z}), w) \to L^{\langle -1 \rangle}_{2k+1}(\mathcal{C}_{W,G}(\mathbf{Z}), w)$$

is injective.

Proof. We first note that the cone point maps $K_0(RG) \to K_0(\mathcal{C}_{W,G}(R))$ are surjective for $R = \widehat{\mathbf{Z}}, \mathbf{Q}$ or $\widehat{\mathbf{Q}}$ since for these coefficients RG has vanishing K_{-1} groups. This shows that $K_{-1}(\mathcal{C}_{W,G}(\mathbf{Z}))$ is again a quotient of $K_0(\widehat{\mathbf{Q}}G)$. To see that $K_{-1}(\mathcal{C}_{W,G}(\mathbf{Z}))$ is also torsion free, consider the boundary map $K_1(\mathcal{C}^{>0}_{W,G}(\widehat{\mathbf{Q}})) \to K_0(\mathcal{C}_W(\widehat{\mathbf{Q}}))$ which is just a sum of induction maps $K_0(\widehat{\mathbf{Q}}K) \to K_0(\widehat{\mathbf{Q}}G)$ from proper subgroups of $K \subset G$. But for G cyclic, these induction maps are split injective. We now complete the argument by comparing the diagram above with the corresponding diagram for the bounded theory, concluding that $H^0(K_{-1}(\mathbf{Z}G)) \to H^0(K_{-1}(\mathcal{C}_{W,G}(\mathbf{Z})))$ is surjective. Since $L^{\langle -1 \rangle}_{2k+2}(\mathbf{Z}G, w) \to H^0(K_{-1}(\mathbf{Z}G))$ is also surjective, we are done. □

7. The proof of Theorem A

The condition (i) is equivalent to assuming that $S(V_1)$ and $S(V_2)$ are freely G–homotopy equivalent. Condition (ii) is necessary by Corollary 3.10 which rules out non–linear similarities of semifree representations. Condition (ii) also implies that $S(V_1)$ is s–normally cobordant to $S(V_2)$ by [3, Prop. 2.1], which is another necessary condition for topological similarity. Thus under conditions (i) and (ii), there exists a homotopy equivalence $f \colon S(V_2) \to S(V_1)$, and an element $\sigma = \sigma(f) \in L^h_0(\mathbf{Z}G)$ such that $trf_{\mathbf{R}_-}(\sigma) = 0 \in L^h_1(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z}))$ if and only if $V_1 \oplus \mathbf{R}_- \sim_t V_2 \oplus \mathbf{R}_-$.

Comparing the h- and s- surgery exact sequences it is easy to see that the image of σ in $H^0(\mathrm{Wh}(\mathbf{Z}G))$ is given by the Whitehead torsion $\{\tau(f)\} = \{\Delta(V_1)/\Delta(V_2)\} \in \mathrm{Wh}(\mathbf{Z}G)$ of the homotopy equivalence $S(V_2)/G \simeq S(V_1)/G$.

In Section 2 we gave the short exact sequence

$$0 \to \mathrm{Wh}(\mathbf{Z}G)/\mathrm{Wh}(\mathbf{Z}H) \to \mathrm{Wh}(\mathbf{Z}H \to \mathbf{Z}G) \to \mathbf{k} \to 0$$

where $K_1(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z}))/\{\pm G\})$ is denoted by $\mathrm{Wh}(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})) = \mathrm{Wh}(\mathbf{Z}H \to \mathbf{Z}G)$ and $\mathbf{k} = \ker(\widetilde{K}_0(\mathbf{Z}H) \to \widetilde{K}_0(\mathbf{Z}G))$. We proved in Theorem 3.6 that the transfer of the torsion element in $\mathrm{Wh}(\mathbf{Z}G)$ in $\mathrm{Wh}(\mathbf{Z}H \to \mathbf{Z}G)$ is given by the same element under the map induced by inclusion $\mathrm{Wh}(\mathbf{Z}G) \to \mathrm{Wh}(\mathbf{Z}H \to \mathbf{Z}G)$.

It follows that the image of $trf_W(\sigma)$ in

$$H^1(\mathrm{Wh}(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z}))) = H^1(\mathrm{Wh}(\mathbf{Z}H \to \mathbf{Z}G^-))$$

is given by the image of our well-defined element

$$\{\Delta(V_1)/\Delta(V_2)\} \in H^1(\mathrm{Wh}(\mathbf{Z}G^-)/\mathrm{Wh}(\mathbf{Z}H))$$

under the cone point inclusion into $H^1(\mathrm{Wh}(\mathbf{Z}H \to \mathbf{Z}G^-))$.

The necessity of the condition is now easy. To have a non-linear similarity we must have

$$trf_{\mathbf{R}_-}(\sigma) = 0 \in L^h_1(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})) \ .$$

Hence

$$trf_{\mathbf{R}_-}(\Delta(V_1)/\Delta(V_2)) = 0 \in H^1(\mathrm{Wh}(\mathbf{Z}H \to \mathbf{Z}G^-))$$

must vanish by naturality of the transfer in the Rothenberg sequence. This element comes from $H^1(\mathrm{Wh}(\mathbf{Z}G^-)/\mathrm{Wh}(\mathbf{Z}H))$, so to vanish in $H^1(\mathrm{Wh}(\mathbf{Z}H \to \mathbf{Z}G^-))$ it must be in the image from

$$H^0(\mathbf{k}) \to H^1(\mathrm{Wh}(\mathbf{Z}G^-/\mathrm{Wh}(\mathbf{Z}H))$$

under the coboundary.

To prove sufficiency, we assume that the image of the transferred element

$$\{\Delta(V_1)/\Delta(V_2)\} \in H^1(\mathrm{Wh}(\mathbf{Z}H \to \mathbf{Z}G^-))$$

is zero. Consider the long exact sequence derived from the inclusion of filtered categories

$$\mathcal{C}_{pt}(\mathbf{Z}G) \subset \mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})$$

where $\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})$ has the standard orientation [**II**], Example 5.4, inducing the non–trivial orientation at the cone point. The quotient category $\mathcal{C}^{>0}_{\mathbf{R}_-,G}(\mathbf{Z})$ of germs away from 0 is canonically isomorphic to $\mathcal{C}^{>0}_{[0,\infty),H}(\mathbf{Z})$, since by equivariance what happens on the positive half line has to be copied on the negative half line, and what happens near 0 does not matter in the germ category. Since the action of H on $[0,\infty)$ is trivial, this category is precisely $\mathcal{C}^{>0}_{[0,\infty)}(\mathbf{Z}H)$ which has the same K- and L-theory as $\mathcal{C}^{>}_{\mathbf{R}}(\mathbf{Z}H)$ by the projection map. By [**II**], Theorem 5.7 we thus get a long exact sequence

$$\ldots \to L^k_n(\mathbf{Z}H) \to L^h_n(\mathbf{Z}G^-) \to L^h_n(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})) \to L^k_{n-1}(\mathbf{Z}H) \to \ldots$$

where the map is induced by induction. Comparing this sequence with the long exact sequence for the pair $L^{h,k}_n(\mathbf{Z}H \to \mathbf{Z}G^-)$ (see Theorem 4.1) it follows that

$$L^h_n(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})) \cong L^{k,h}_n(\mathbf{Z}H \to \mathbf{Z}G^-) \ .$$

Consider the following diagram with exact rows and columns:

$$\begin{array}{ccccc}
L_1^{\mathrm{Wh}(\mathbf{Z}H)}(\mathbf{Z}G^-) & \longrightarrow & L_1^{h,\mathrm{Wh}(\mathbf{Z}H)}(\mathbf{Z}H \to \mathbf{Z}G^-) & \longrightarrow & L_0^h(\mathbf{Z}H) \\
\downarrow & & \downarrow & & \downarrow \\
L_1^h(\mathbf{Z}G^-) & \longrightarrow & L_1^{k,h}(\mathbf{Z}H \to \mathbf{Z}G^-) & \longrightarrow & L_0^k(\mathbf{Z}H) \\
\downarrow & & \downarrow & & \downarrow \\
H^1(\mathrm{Wh}(G^-)/\,\mathrm{Wh}(H)) & \longrightarrow & H^1(\mathrm{Wh}(\mathbf{Z}H \to \mathbf{Z}G^-)) & \longrightarrow & H^1(\mathbf{k})
\end{array}$$

We need to show that $\mathit{trf}_{\mathbf{R}_-}(\sigma)$ vanishes in order to produce the non-linear similarity. We know that the image of $\mathrm{Res}_H(\sigma) = 0$, and our assumption is that the image

$$\{\Delta(V_1)/\Delta(V_2)\} \in H^1(\mathrm{Wh}(\mathbf{Z}H \to \mathbf{Z}G^-))$$

vanishes. We will finish the argument by showing:

Lemma 7.1. *Suppose that $\sigma \in L_0^h(\mathbf{Z}G)$.*

(i) *$L_1^{h,\mathrm{Wh}(\mathbf{Z}H)}(\mathbf{Z}H \to \mathbf{Z}G^-)$ is torsion–free.*

(ii) *The torsion subgroup of $L_1^{k,h}(\mathbf{Z}H \to \mathbf{Z}G^-)$ injects into $H^1(\mathrm{Wh}(\mathbf{Z}H \to \mathbf{Z}G^-))$, or equivalently, the torsion subgroup of $L_1^h(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z}))$ injects into $H^1(\mathrm{Wh}(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})))$.*

(iii) *If $\mathrm{Res}_H(\sigma) = 0 \in L_0^h(\mathbf{Z}H)$, then $\mathit{trf}_{\mathbf{R}_-}(\sigma)$ is a torsion element.*

The proof of Lemma 7.1. For assertion (i), we consider the diagram

$$\begin{array}{ccc}
L_0^s(\mathbf{Z}H) & \longrightarrow & L_0^s(\mathbf{Z}G^-) \\
\downarrow & & \downarrow \\
L_0^h(\mathbf{Z}H) & \longrightarrow & L_0^{\mathrm{Wh}(\mathbf{Z}H)}(\mathbf{Z}G^-) \\
\downarrow & & \downarrow \\
H^0(\mathrm{Wh}(\mathbf{Z}H)) & = & H^0(\mathrm{Wh}(\mathbf{Z}H))
\end{array}$$

where $L_0^s(\mathbf{Z}G^-) = 0$. Since $L_1^{\mathrm{Wh}(\mathbf{Z}H)}(\mathbf{Z}G^-) = 0$ as well (by Lemma 6.11), it follows that

$$L_1^{h,\mathrm{Wh}(\mathbf{Z}H)}(\mathbf{Z}H \to \mathbf{Z}G^-) = \ker(L_0^h(\mathbf{Z}H) \to L_0^{\mathrm{Wh}(\mathbf{Z}H)}(\mathbf{Z}G^-)) \cong L_0^s(\mathbf{Z}H)$$

where $L_0^s(\mathbf{Z}H)$ is torsion–free. Part (ii) follows from part (i), since the previous term in the exact sequence has exponent two. For assertion (iii) we refer to part of the twisting diagram of Section 4, namely the commutative diagram:

$$\begin{array}{ccccc}
 & & L_0^h(\mathbf{Z}G) & & \\
 & & \downarrow \mathit{trf}_{\mathbf{R}_-} & \searrow \mathrm{Res}_H & \\
L_1^h(\mathbf{Z}G^-) & \longrightarrow & L_1^{k,h}(\mathbf{Z}H \to \mathbf{Z}G^-) & \longrightarrow & L_0^k(\mathbf{Z}H)
\end{array}$$

and the fact that $\mathrm{Res}_H(\mathit{trf}_{\mathbf{R}_-}(\sigma)) = 0$ since it factors through $\mathrm{Res}_H\colon L_0^h(\mathbf{Z}G) \to L_0^h(\mathbf{Z}H)$. But $L_1^h(\mathbf{Z}G^-) \cong H^1(\mathrm{Wh}(\mathbf{Z}G^-)$ has exponent 2. □

Remark 7.2. Theorem A gives necessary and sufficient conditions for the existence of 5–dimensional similarities. Consider the situation in dimensions $\leqq 5$. By character theory it suffices to consider a cyclic group G, which must be of order divisable by 4, say $4q$, with index 2 subgroup H. It suffices by Lemma 3.1 to consider the following situation

$$V_1 \oplus W \sim_t V_2 \oplus W$$

where V_i are free homotopy equivalent representations which become isomorphic once restricted to H. Therefore $S(V_1)$ is normally cobordant to $S(V_2)$. If V_i are two-dimensional they are determined by one character, so homotopy equivalence implies isomorphism. We may thus assume V_i are at least 4-dimensional, and $\dim W = 1$. According to Theorem 3.5, non-linear similarity is now determined by $trf_W(\sigma)$, where σ is an element of infinite order in $L^h_0(\mathbf{Z}G)$ hitting the element in the structure set determined by the homotopy equivalence of $S(V_2)/G \simeq S(V_1)/G$. In case W is the trivial representation we may identify trf_W with the map

$$L^h_0(\mathbf{Z}G) \to L^p_0(\mathbf{Z}G)$$

which we have seen (in Corollary 3.10) is injective on the elements of infinite order. Hence we are left with the case where W is the non-trivial one-dimensional representation $\mathbf{R}_-$, which is reduced to number theory by Theorem A. In Corollary 9.3 we work out the number theory for $G = C(2^r)$ as an example, showing that 5–dimensional similarities do not exist for these groups. The general case was done by Cappell and Shaneson in 1981, and this preprint has recently been published [6].

8. The proof of Theorem B

The new ingredient in Theorem B is the double coboundary. As in the last section, we may assume that $f\colon S(V_2) \to S(V_1)$ is a G–homotopy equivalence which is freely G–normally cobordant to the identity, giving an element $\sigma = \sigma(f) \in L^h_0(\mathbf{Z}G)$. Then $V_1 \oplus \mathbf{R}_- \oplus \mathbf{R}_+ \sim_t V_2 \oplus \mathbf{R}_- \oplus \mathbf{R}_+$ if and only if

$$trf_{\mathbf{R}_-\oplus\mathbf{R}_+}(\sigma) = 0 \in L^h_2(\mathcal{C}_{\mathbf{R}_-\oplus\mathbf{R}_+,G}(\mathbf{Z}))$$

by Theorem 3.5. But $L^h_2(\mathcal{C}_{\mathbf{R}_-\oplus\mathbf{R}_+,G}(\mathbf{Z})) = L^p_1(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z}))$, so we may regard the criterion as the vanishing of $trf_{\mathbf{R}_-}(\sigma) \in L^p_1(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z}))$ instead.

The main commutative diagram is:

(8.1)
$$\begin{array}{ccccc}
H^1(\widetilde{K}_0(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z}))) & \xrightarrow{\delta^2} & H^1(\mathrm{Wh}(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z}))) & \longrightarrow & L^s_0(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})) \\
& \searrow \quad \nearrow & & \searrow \quad \nearrow & \\
& L^h_1(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})) & & H^1(\Delta_{\mathbf{R}_-}) & \\
& \nearrow \quad \searrow & & \nearrow \quad \searrow & \\
L^s_1(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})) & \longrightarrow & L^p_1(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})) & \longrightarrow & H^0(\widetilde{K}_0(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})))
\end{array}$$

where $H^1(\Delta_{\mathbf{R}_-})$ denotes the relative group of the double coboundary map. Notice that some of the groups in this diagram already appeared as relative L–groups in the last section. We have

$$L_1^h(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})) = L_1^{k,h}(\mathbf{Z}H \to \mathbf{Z}G^-)$$

and

$$L_1^s(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})) = L_1^{h,\mathrm{Wh}(\mathbf{Z}H)}(\mathbf{Z}H \to \mathbf{Z}G^-)\ .$$

By Lemma 7.1 (i) this group is torsion–free and the preceeding term $H^0(\Delta_{\mathbf{R}_-})$ is 2-torsion. Hence $L_1^s(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z}))$ injects into $L_1^p(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z}))$, and we conclude that the torsion subgroup of $L_1^p(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z}))$ injects into $H^1(\Delta_{\mathbf{R}_-})$.

However, since $\mathrm{Res}_H(\sigma) = 0 \in L_0^h(\mathbf{Z}H)$, Lemma 7.1 (iii) states that $\mathit{trf}_{\mathbf{R}_-}(\sigma)$ is a torsion element. Furthermore, its image in $H^1(\Delta_{\mathbf{R}_-})$ is zero if and only if

$$\{\Delta(V_1)/\Delta(V_2)\} \in H^1(\mathrm{Wh}(\mathbf{Z}H \to \mathbf{Z}G^-))$$

lies in the image of the double coboundary, and this completes the proof of Theorem B.

We conclude this section with an important property of the relative group $H^1(\Delta_{W\times\mathbf{R}_-})$ of the double coboundary

$$\delta^2\colon H^1(\widetilde{K}_0(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))) \to H^1(\mathrm{Wh}(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z})))$$

which we will refer to as *Galois invariance*. For G a cyclic 2-group, we establish a similar statement for the image of $H^1(\mathrm{Wh}(\mathbf{Z}G^-)/\mathrm{Wh}(\mathbf{Z}H))$ in the relative group $H^1(\Delta_{\mathbf{R}_-})$. This sharper version will be used in determining the non-linear similarities for cyclic 2-groups.

Lemma 8.2. *Let W be a complex G–representation, with $W^G = 0$, containing all the non-trivial irreducible representations of G with isotropy of 2-power index.*

(i) *Automorphisms of G induce the identity on the image of $H^1(\mathrm{Wh}(\mathbf{Z}G^-)/\mathrm{Wh}(\mathbf{Z}H))$ in the relative group $H^1(\Delta_{W\times\mathbf{R}_-})$ of the double coboundary.*
(ii) *If G is a cyclic 2-group, then automorphisms of G induce the identity on the image of $H^1(\mathrm{Wh}(\mathbf{Z}G^-)/\mathrm{Wh}(\mathbf{Z}H))$ in the relative group $H^1(\Delta_{\mathbf{R}_-})$.*

Proof. The first step is to prove that the image of any $[u] \in H^1(\mathrm{Wh}(\mathbf{Z}G^-)/\mathrm{Wh}(\mathbf{Z}H))$ in $H^1(\mathrm{Wh}(\mathbf{Z}H \to \mathbf{Z}G^-)) = H^1(\mathrm{Wh}(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})))$ equals the discriminant

$$\mathit{trf}_{\mathbf{R}_-}(\sigma) \in L_1^h(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z}))$$

for some $\sigma \in L_0^h(\mathbf{Z}G)$. Consider the diagram

$$\begin{array}{ccc} L_0^h(\mathbf{Z}G) & \longrightarrow & H^0(\mathrm{Wh}(\mathbf{Z}G)) \\ \downarrow{\scriptstyle \mathit{trf}_{\mathbf{R}_-}} & & \downarrow{\scriptstyle \mathit{trf}_{\mathbf{R}_-}} \\ L_1^h(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})) & \longrightarrow & H^1(\mathrm{Wh}(\mathbf{Z}H \to \mathbf{Z}G^-)) \end{array}$$

By Theorem 3.6, the K-theory transfer factors through the map induced by the cone point inclusion $c_*\colon H^1(\mathrm{Wh}(\mathbf{Z}G^-)/\mathrm{Wh}(\mathbf{Z}H)) \to H^1(\mathrm{Wh}(\mathbf{Z}H \to \mathbf{Z}G^-))$. Since $L_0^h(\mathbf{Z}G) \to H^0(\mathrm{Wh}(\mathbf{Z}G)$ is surjective, there exists $\sigma \in L_0^h(\mathbf{Z}G)$ with given discriminant $[u]$, and $\mathit{trf}_{\mathbf{R}_-}(\sigma)$

has discriminant $c_*([u])$. But the natural map $H^0(\mathrm{Wh}(\mathbf{Z}G)) \to H^1(\mathrm{Wh}(\mathbf{Z}G^-)/\mathrm{Wh}(\mathbf{Z}H))$ is also surjective, so the first step is complete. By naturality of the transfer

$$trf_W\colon L^h_1(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})) \to L^h_{2k+1}(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))$$

($\dim W = 2k$), it follows that the image of $c_*([u])$ in $H^1(\mathrm{Wh}(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z})))$ equals the discriminant of the element $trf_{W\times\mathbf{R}_-}(\sigma)$.

Our assumption on W implies that $trf_{W\times\mathbf{R}_-}(\sigma)$ is a torsion element (see [**II**], Section 4). We can therefore apply [**II**], Theorem 8.1(i): there exists a *torsion* element $\hat\sigma \in L^h_{2k+1}(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z}))$, such that $c_*(\hat\sigma) = trf_{W\times\mathbf{R}_-}(\sigma)$. Now from the commutative diagram comparing cone point inclusions:

$$\begin{array}{ccc} L^h_{2k+1}(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})) & \longrightarrow & H^1(\mathrm{Wh}(\mathbf{Z}H\to\mathbf{Z}G^-)) \\ \downarrow{\scriptstyle c_*} & & \downarrow{\scriptstyle c_*} \\ L^h_{2k+1}(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z})) & \longrightarrow & H^1(\mathrm{Wh}(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z})) \end{array}$$

we conclude that the image of $c_*([u])$ in $H^1(\mathrm{Wh}(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z})))$ equals the image of the discriminant of $\hat\sigma \in L^h_{2k+1}(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z}))$. The braid diagram used in the proof of Theorem B above now shows that the image of $c_*([u])$ in $H^1(\Delta_{W\times\mathbf{R}_-})$ comes from the image of $\hat\sigma \in L^p_{2k+1}(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z}))$ in $H^1(\Delta_{\mathbf{R}_-})$, via the natural map $H^1(\Delta_{\mathbf{R}_-}) \to H^1(\Delta_{W\times\mathbf{R}_-})$.

Finally we consider the cone point inclusion sequence

$$L^h_2(\mathcal{C}^{\geq 0}_{\mathbf{R}_-,G}(\mathbf{Z})) \to L^p_1(\mathbf{Z}G^-) \to L^p_1(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})) \to L^h_1(\mathcal{C}^{\geq 0}_{\mathbf{R}_-,G}(\mathbf{Z})) \to L^p_0(\mathbf{Z}G^-)$$

The K–theory decoration on the relative group is the image of

$$\widetilde{K}_0(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})) \to \widetilde{K}_0(\mathcal{C}^{\geq 0}_{\mathbf{R}_-,G}(\mathbf{Z})) \cong K_{-1}(\mathbf{Z}H)$$

but since $K_{-1}(\mathbf{Z}H) \to K_{-1}(\mathbf{Z}G)$ is injective, that image is zero and we get L^h. Now $L^h_2(\mathcal{C}^{\geq 0}_{\mathbf{R}_-,G}(\mathbf{Z})) \cong L^p_1(\mathbf{Z}H) = 0$ and $L^h_1(\mathcal{C}^{\geq 0}_{\mathbf{R}_-,G}(\mathbf{Z})) \cong L^p_0(\mathbf{Z}H)$ is torsion–free. It follows that the cone point inclusion $L^p_1(\mathbf{Z}G^-) \to L^p_1(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z}))$ is an isomorphism onto the torsion subgroup. But by Corollary 6.3 the group $L^p_1(\mathbf{Z}G^-)$ is fixed by group automorphisms of G. This completes the proof of part (i).

In part (ii) we assume that G is a cyclic 2-group, so $K_{-1}(\mathbf{Z}K) = 0$ for all subgroups $K \subseteq G$. By [**II**], Corollary 6.9 we have an isomorphism $H^1(\Delta_{\mathbf{R}_-}) \cong H^1(\Delta_{W\times\mathbf{R}_-})$ and the proof is complete. □

9. Cyclic 2–Groups: Preliminary Results

For $G = C(2^r)$ a cyclic 2–group, we have stronger results because $K_{-1}(\mathbf{Z}G) = 0$. The results of this section prepare for a complete classification of stable and unstable non–linear similarities for cyclic 2–groups, and the computation of $R_{\mathrm{Top}}(G)$.

Lemma 9.1. (R. Oliver, [24]) *For $G = C(2^r)$, the cohomology groups $H^*(\mathbf{k}) = 0$, where $\mathbf{k} = \ker(\widetilde{K}_0(\mathbf{Z}H) \to \widetilde{K}_0(\mathbf{Z}G))$.*

Proof. We are indebted to R. Oliver for pointing out that this result follows from [24, Thm. 2.6], which states (in his notation):

$$D(Z[C(2^{n+3})) \cong \mathrm{Im}((5-\gamma)\hat\psi_n)$$

where $D(\mathbf{Z}G)$ is the kernel of the map induced by including $\mathbf{Z}G$ in a maximal order $\mathcal{M}$ in $\mathbf{Q}G$. By Weber's Theorem, the ideal class groups of 2-power cyclotomic fields have odd order, so to prove that $H^*(\mathbf{k}) = 0$ it is enough to show that $\ker(D(\mathbf{Z}H) \to D(\mathbf{Z}G)) = 0$.

The map $\hat{\psi}$ is the reduction mod 2^{n+1} of a map $\psi_n\colon M^n \to M^n$ given by the formula

$$\psi_n(e_i) = \sum_{j=0}^{i} 2^{i-j}\gamma_{i-j}^{n-j} e_{n-j}$$

for $0 \le i \le n$, where $\{e_0, \dots, e_n\}$ is a basis for the direct sum $M^n = \sum_{i=0}^n \widehat{\mathbf{Z}}_2[\Gamma_i]e_i$ and Γ_i denotes the cyclic group of automorphisms of $C(2^{i+2})$ generated by $\gamma(t) = t^5$.

To shorten Oliver's notation, we let $D[n+3] := D(Z[C(2^{n+3}))$ so $D[n+3]$ is identified with the subgroup $\operatorname{Im}((5-\gamma)\hat{\psi}_n)$ of $M^n/2^{n+1}M^n$. It is enough to see that the map

$$\operatorname{Ind}\colon D[n+3] \to D[n+4]$$

given by the subgroup inclusion induces an injection on this subgroup $\operatorname{Im}((5-\gamma)\hat{\psi}_n)$.

However the map Ind corresponds under the identification in [24, Thm. 2.4] with the explicit map $\operatorname{ind}(e_i) = \gamma_i^{i+1}e_{i+1}$ (see last paragraph of [24, §2]). Using this explicit formula, we need to check that $x \in \ker(\hat{\psi}_{n+1} \circ \operatorname{ind})$ implies that $x \in \ker \hat{\psi}_n$. Suppose that $x = \sum_{i=0}^n a_i e_i \in M^n$. Then

$$\psi_{n+1}(\operatorname{ind}(x)) = \left(\sum_{i=0}^{n} a_i 2^{i+1}\gamma_i^{i+1}\gamma_{i+1}^{n+1}\right) e_{n+1} + \sum_{j=0}^{n}\left(\sum_{i=j}^{n} a_i 2^{i-j}\gamma_i^{i+1}\gamma_{i-j}^{n-j}\right) e_{n-j}$$

after re-arranging the summations, and

$$\psi_n(x) = \sum_{j=0}^{n}\left(\sum_{i=j}^{n} a_i 2^{i-j}\gamma_{i-j}^{n-j}\right) e_{n-j} \ .$$

We can then use the formulas after [24, Lemma 1.1] to check that

$$\gamma_i^{i+1}\gamma_{i-j}^{n-j} = 2\gamma_{i-j}^{n-j} \in \widehat{\mathbf{Z}}_2[\Gamma_{n-j}]$$

for $0 \leqq j \leqq n$. Since $M^{n+1}/2^{n+2}M^{n+1}$ is a direct sum of the group rings $\mathbf{Z}/2^{n+2}[\Gamma_i]$, it follows that $\psi_{n+1}(\operatorname{ind}(x)) \equiv 0 \bmod 2^{n+2}$ implies $\psi_n(x) \equiv 0 \bmod 2^{n+1}$. $\square$

Our results for cyclic 2-groups can now be improved, starting with similarities with an $\mathbf{R}_-$ but no $\mathbf{R}_+$ summand.

Theorem 9.2. *Let $V_1 = t^{a_1} + \dots + t^{a_k}$ and $V_2 = t^{b_1} + \dots + t^{b_k}$ be free G–representations, where $G = C(2^r)$. Let W be a complex G–representation with no $\mathbf{R}_+$ or $\mathbf{R}_-$ summands.*

(i) *If $\operatorname{Res}_H(V_1 \oplus W) \oplus \mathbf{R}_+ \sim_t \operatorname{Res}_H(V_2 \oplus W) \oplus \mathbf{R}_+$, then $V_1 \oplus W \oplus \mathbf{R}_- \sim_t V_2 \oplus W \oplus \mathbf{R}_-$ if and only if $S(V_1)$ is s–normally cobordant to $S(V_2)$ and $\{\Delta(V_1)/\Delta(V_2)\} = 0$ in $H^1(\operatorname{Wh}(\mathbf{Z}G^-)/\operatorname{Wh}(\mathbf{Z}H))$.*

(ii) *If $\operatorname{Res}_H V_1 \cong \operatorname{Res}_H V_2$, then $V_1 \oplus W \oplus \mathbf{R}_- \sim_t V_2 \oplus W \oplus \mathbf{R}_-$ if and only if the class $\{\Delta(V_1)/\Delta(V_2)\} = 0$ in $H^1(\operatorname{Wh}(\mathbf{Z}G^-))$.*

Proof. Consider first the situation in part (i). By [**II**], Theorem 9.2, the image of the final surgery obstruction $trf_{W\times\mathbf{R}_-}(\sigma)$ in $L^h_1(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))$ is just the class $\{\Delta(V_1)/\Delta(V_2)\}$ considered as an element in $H^1(\mathrm{Wh}(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))$. Moreover, the natural map

$$H^1(\mathrm{Wh}(\mathbf{Z}G^-)/\,\mathrm{Wh}(\mathbf{Z}H)) \to H^1(\mathrm{Wh}(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))$$

factors through $H^1(\mathrm{Wh}(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})))$, and the torsion subgroup of $L^h_1(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z}))$ injects into $H^1(\mathrm{Wh}(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})))$ by Lemma 7.1(ii). Therefore, $trf_{W\times\mathbf{R}_-}(\sigma) = 0$ if and only if the Reidemeister torsion invariant vanishes in $H^1(\mathrm{Wh}(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})))$. But $H^*(\mathbf{k}) = 0$ so the natural map

$$H^1(\mathrm{Wh}(\mathbf{Z}G^-)/\,\mathrm{Wh}(\mathbf{Z}H)) \to H^1(\mathrm{Wh}(\mathbf{Z}H\to\mathbf{Z}G^-)) \cong H^1(\mathrm{Wh}(\mathcal{C}_{\mathbf{R}_-,G}(\mathbf{Z})))$$

induces an isomorphism. This proves part (i).

In part (ii), since $\mathrm{Res}_H V_1 \cong \mathrm{Res}_H V_2$, our Reidemeister torsion quotient represents an element $\{\Delta(V_1)/\Delta(V_2)\} \in H^1(\mathrm{Wh}(\mathbf{Z}G^-))$, and this group injects into $H^1(\mathrm{Wh}(\mathbf{Z}H\to\mathbf{Z}G^-))$. The vanishing of the surgery obstruction is now equivalent to $\{\Delta(V_1)/\Delta(V_2)\} = 0$ in $H^1(\mathrm{Wh}(\mathbf{Z}G^-))$, by the argument above. □

Corollary 9.3. *The groups $G = C(2^r)$ have no 5–dimensional non–linear similarities.*

Proof. The Reidemeister torsion quotients for possible 5-dimensional similarities are represented by the units $U_{1,i}$ which form a basis of $H^1(\mathrm{Wh}(\mathbf{Z}G^-))$ (see [5], [8, p.733]). □

Higher–dimensional similarities of cyclic 2–groups were previously studied in the 1980's. The 6–dimensional case was worked out in detail for cyclic 2–groups in [8], and general conditions *A–D* were announced in [7] for the classification of non–linear similarities for cyclic 2-groups in any dimension. However, in Example 9.7 we give a counterexample to the necessity of [7, Condition B], and this invalidates the claimed solution. Our next result concerns similarities with both $\mathbf{R}_-$ and $\mathbf{R}_+$ summands.

Theorem 9.4. *Let $V_1 = t^{a_1} + \cdots + t^{a_k}$ and $V_2 = t^{b_1} + \cdots + t^{b_k}$ be free G–representations, where $G = C(2^r)$. Let W be a complex G–representation with no $\mathbf{R}_+$ summands. Then there exists a topological similarity $V_1 \oplus W \oplus \mathbf{R}_- \oplus \mathbf{R}_+ \sim_t V_2 \oplus W \oplus \mathbf{R}_- \oplus \mathbf{R}_+$ if and only if*

(i) *$S(V_1)$ is s–normally cobordant to $S(V_2)$,*
(ii) $\mathrm{Res}_H(V_1 \oplus W) \oplus \mathbf{R}_+ \sim_t \mathrm{Res}_H(V_2 \oplus W) \oplus \mathbf{R}_+$, *and*
(iii) *the element $\{\Delta(V_1)/\Delta(V_2)\}$ is in the image of the double coboundary*

$$\delta^2 \colon H^1(\widetilde{K}_0(\mathbf{Z}H\to\mathbf{Z}G^-)) \to H^1(\mathrm{Wh}(\mathbf{Z}H\to\mathbf{Z}G^-))\ .$$

Corollary 9.5. *There is a stable topological similarity $V_1 \approx_t V_2$ if and only if $S(V_1)$ is s–normally cobordant to $S(V_2)$ and $\mathrm{Res}_K\{\Delta(V_1)/\Delta(V_2)\}$ is in the image of the double coboundary δ^2_K for all subgroups $K \subseteq G$.*

Remark 9.6. The double coboundary δ^2_K in this statement is the one for the subgroup K with respect to an index two subgroup $K_1 < K$. Corollary 9.5 follows from Theorem 9.4 and [**II**], Proposition 7.6. Notice that Theorem 9.4 also gives a way to construct the stable similarity, assuming that the conditions are satisfied. In the most complicated case, one would need all proper subgroups of G appearing as isotropy groups in W. This will be explained precisely in Section 11.

The proof of Theorem 9.4. By [**II**], Corollary 6.9 the double coboundary

$$\delta^2_W\colon H^1(\widetilde{K}_0(\mathcal{C}_{W_{max}\times\mathbf{R}_-,G}(\mathbf{Z}))) \to H^1(\mathrm{Wh}(\mathcal{C}_{W_{max}\times\mathbf{R}_-,G}(\mathbf{Z})))$$

is isomorphic to

$$\delta^2\colon H^1(\widetilde{K}_0(\mathbf{Z}H \to \mathbf{Z}G^-)) \to H^1(\mathrm{Wh}(\mathbf{Z}H \to \mathbf{Z}G^-))$$

under the cone point inclusion. The result now follows from Theorem C. □

We conclude this section with two examples showing methods of constructing non–linear similarities.

Example 9.7. For $G = C(2^r)$, $r \geqq 5$, the element $(t^9 + t^{1+2^{r-2}} - t - t^{9+2^{r-2}})$ lies in $\widetilde{R}^{free}_t(G)$. To prove this assertion, note that $(t^9 + t^{1+2^{r-2}} - t - t^{9+2^{r-2}})$ lies in $\widetilde{R}^{free}_n(G)$ by applying [7, Condition A′]. Let $V_1 = t^9 + t^{1+2^{r-2}}$, $V_2 = t + t^{9+2^{r-2}}$ and let W denote the complex 2–dimensional representation with isotropy of index 4. The surgery obstruction in $L^p_3(\mathcal{C}_{W\times\mathbf{R}_-,G}(\mathbf{Z}))$ is zero by a similar argument to that given in [8, p.734], based on the fact that $9(1+2^{r-2}) \equiv 1 \bmod 8$ when $r \geqq 5$, and invariance under group automorphisms (similar to [8, 4.1]). The Reidemeister torsion invariant

$$u = \Delta(V_1)/\Delta(V_2) = \frac{(t^9-1)(t^{1+2^{r-2}}-1)}{(t-1)(t^{9+2^{r-2}}-1)}$$

can be written in the form $u = \alpha(v)v$ where

$$v = \frac{(t^\ell-1)(t^{1+2^{r-2}}-1)}{(t-1)(t^{\ell+2^{r-2}}-1)}$$

for $\ell^2 \equiv 9 \bmod 2^r$ and $\ell \equiv 1 \bmod 4$. The Galois automorphism is $\alpha(t) = t^\ell$. Once again we justify these formulas by appeal to a pull–back diagram for the group ring. Since the surgery obstruction is determined by the image of u in $H^1(\Delta_{\mathbf{R}_-})$, which has exponent 2, and group automorphisms of G induce the identity on this group by Lemma 8.2(ii), the element $(t^9 + t^{1+2^{r-2}} - t - t^{9+2^{r-2}})$ lies in $\widetilde{R}^{free}_t(G)$.

We note that this gives a counterexample to the *necessity* of [7, Condition B] for non-linear similarities. If $f\colon S(V_1) \to S(V_2)$ is a G-homotopy equivalence, then the Whitehead torsion $\tau(f) = \Delta(V_1)/\Delta(V_2) \in \mathrm{Wh}(\mathbf{Z}G)$, but its restriction to $\mathrm{Wh}(\mathbf{Z}H)$ is given by $U_{9,1+2^{r-2}} = U_{1,9}(U_{1,1})^{-1}$, and this is *not* a square, since it is non–trivial in $H^0(\mathrm{Wh}(\mathbf{Z}H))$ by the results of Cappell–Shaneson on units (see [8, p.733]).

Example 9.8. According to [7, Thm. 2], the element $t - t^5$ has order 2^{r-2} in $R_{\mathrm{Top}}(G)$ for $G = C(2^r)$, $r > 3$, and order 4 for $G = C(8)$. We will verify this claim using our methods. Note that $2^{r-2}(t-t^5) \in \widetilde{R}^{free}_h(G)$, for $r \geqq 3$ and this is the smallest multiple that works. A short calculation using [7, Condition A′] (done in [**II**], Lemma 12.5) also shows that $2^{r-2}(t-t^5) \in \widetilde{R}^{free}_n(G)$, and it remains to consider the surgery obstruction.

For $r = 3$ it follows from [8, Cor.(iii)] that $2(t - t^5)$ does not give a 6–dimensional similarity. By Theorem 9.4 it follows that $2(t - t^5) \notin R_t(G)$, but is contained in $R_n(G)$. Since the surgery obstruction has exponent 2, we get $4(t - t^5) \approx_t 0$ as claimed.

To handle the general case, let $U_1 = 2^{r-3}t$ and $U_2 = 2^{r-3}t^5$ be the free representations over $H = C(2^{r-1})$, and let $V_1 = 2^{r-3}t$ and $V_2 = 2^{r-3}t^5$ be the corresponding free representations over $G = C(2^r)$. Notice that $\mathrm{Ind}_H(U_1-U_2) = (V_1-V_2)+(V_1^\tau-V_2^\tau)$ where τ is the group automorphism $\tau(t) = t^{1+2^{r-1}}$. By [8, Cor(iii)] we have $2t \approx_t 2t^{1+2^{r-1}}$ and $2t^5 \approx_t 2t^{5+2^{r-1}}$, whenever $r \geqq 4$. Therefore, $2^{r-2}t = V_1 \oplus V_1 \approx_t V_1 \oplus V_1^\tau$ and $2^{r-2}t^5 = V_2 \oplus V_2 \approx_t V_2 \oplus V_2^\tau$.

For $r > 3$, the assertion is that $2^{r-2}(t - t^5) \in \widetilde{R}_t^{free}(G)$ already. We prove this by induction starting with $2(t - t^5)$ in $R_n(C(8))$. Suppose that $2^{r-3}(t - t^5) \in R_n(H)$ for $H = C(2^{r-1})$, where $r > 3$. Then $\operatorname{Ind}_H(\Delta(U_1)/\Delta(U_2)) = 0$ in $H^1(\operatorname{Wh}(\mathbf{Z}G^-)/\operatorname{Wh}(\mathbf{Z}H))$, since we are dividing out exactly the image of Ind_H. If $r = 4$, it follows from Theorem 9.4 that $\operatorname{Ind}_H(U_1 - U_2) \in R_t(C(16))$. For $r \geqq 5$ we conclude by induction that $\operatorname{Ind}_H(U_1 - U_2) \in R_t(G)$, for $G = C(2^r)$ since $R_t(G)$ is closed under induction from subgroups. Now the calculation above for $\operatorname{Ind}_H(U_1 - U_2)$ shows that $2^{r-2}(t - t^5) \in \widetilde{R}_t^{free}(G)$ for $r \geqq 4$.

10. The proof of Theorem E

Our next example is the computation of $R_{\mathrm{Top}}(G)$ for $G = C(2^r)$. We first choose a nice basis for $\widetilde{R}^{free}(G)$. Let

$$a_s^{(i)}(r) = t^{5^i} - t^{5^{2^{r-s-2}+i}}, \quad \text{for } 0 \leqq i < 2^{r-s-2} \text{ and } 1 \leqq s \leqq r-2$$

and let σ denote the automorphism of G given by $\sigma(t) = t^5$. It is easy to check that the $\{a_s^{(i)}(r)\}$ give an additive basis for $\widetilde{R}^{free}(G)$. For later use, we let $\alpha_s(r) = [a_s^{(0)}(r)]$ and $\beta_s(r) = [a_s^{(1)}(r)]$ denote elements in $\widetilde{R}_{\mathrm{Top}}^{free}(G)$ for $1 \leqq s \leqq r-2$. When the order 2^r of G is understood, we will just write α_s, β_s. These elements admit some stable non–linear similarities, and behave well under induction and restriction.

Lemma 10.1. *We have the following relations.*

(i) $\operatorname{Res}_H(a_s^{(i)}(r)) = a_{s-1}^{(i)}(r-1)$ *for* $0 \leqq i < 2^{r-s-2}$ *and* $s \geqq 2$.
(ii) $\operatorname{Res}_H(a_1^{(i)}(r)) = 0$.
(iii) $a_s^{(i)}(r) \approx_t a_s^{(i+2)}(r)$ *for* $0 \leqq i < 2^{r-s-2} - 2$, *with* $1 \leqq s \leqq r-4$ *and* $r \geqq 5$.
(iv) $0 \neq \alpha_1(r) + \beta_1(r) \in \widetilde{R}_{n,\mathrm{Top}}^{free}(G)$ *for* $r \geqq 4$, *but* $2(\alpha_1(r) + \beta_1(r)) = 0$.
(v) $\operatorname{Ind}_H(a_{s-1}^{(i)}(r-1)) = 2a_s^{(i)}(r) - a_1^{(i)}(r) + a_1^{(2^{r-s-2}+i)}(r)$ *for* $2 \leqq s \leqq r-2$.
(vi) $2^{s-2}(\alpha_s(r) - \beta_s(r)) \notin \widetilde{R}_{n,\mathrm{Top}}^{free}(G)$ *for* $s \geqq 2$.
(vii) $2^{s-1}(\alpha_s(r) + \beta_s(r)) = 0$ *for* $2 \leqq s < r-2$ *and* $r \geqq 5$.
(viii) $2^s(\alpha_s(r)) = 0$ *for* $1 \leqq s \leqq r-2$ *and* $r \geqq 4$.

Proof. The first two parts are immediate from the definitions. Part (iii) uses the Cappell–Shaneson trick described in Example 9.7. Let

$$u(a,b;c,d) = \frac{(t^{5^a} - 1)(t^{5^b} - 1)}{(t^{5^c} - 1)(t^{5^d} - 1)}$$

This element represents a unit in $\mathbf{Z}G$ provided that $a + b \equiv c + d \bmod 2^{r-2}$. As mentioned in Section 2, we can calculate in a pull–back square for $\mathbf{Z}G$ over the corner where these elements become cyclotomic units. Notice that $u(a,b;c,d)^{-1} = u(c,d;a,b)$ and $\sigma(u(a,b;c,d) = u(a+1, b+1; c+1, d+1)$. Consider the units $u = u(i, 2^{r-s-2} + i + 2; 2^{r-s-2} + i, i + 2)$ associated to the Reidemeister torsion quotient $\Delta(V_1)/\Delta(V_2)$ for the element $a_s^{(i)}(r) - a_s^{(i+2)}(r)$. We can write $u = \sigma(v)v$ where $v = u(i, 2^{r-s-2} + i + 1; i + 1, 2^{r-s-2} + i)$ also represents a unit in $\mathbf{Z}G$. Since $\dim V_i = 4$ the spheres $S(V_1)$ and $S(V_2)$ are G–normally cobordant. Then we restrict to H and use induction on r, starting with $r = 4$ where the similarity follows from [8, Cor.(iii),p.719]. Theorem 9.4 and Galois invariance of the surgery obstruction under the action of σ (by Lemma 8.2(ii)) completes the inductive step.

The non–existence of a 6–dimensional similarity in Part(iv) follows from the calculation [8, Cor.(iii),p.719]. Theorem 9.4 shows that there is no higher dimensional similarity.

Part (v) is again immediate, and Part (vi) is an easy calculation showing that the element $2^{s-2}(\alpha_s(r) - \beta_s(r)) \in \widetilde{R}_h^{free}(G)$ fails the first congruence condition in [34, Thm.1.2], which for this case is the same as the congruence on the sum of the squares of the weights given in [7, Condition A′]. Part (viii) follows from part (v) and induction on the order of G. In Example 9.8 we did the case $s = r - 2$. For $s \leqq r - 3$ we induce up from the similarity $2\alpha_1(r-s+1) = 0$ provided by [8, p.719], which applies since we now have $r-s+1 \geqq 4$. Part (vii) is proved in a similar way, using part (v), starting from the element $\alpha_1(r-1)+\beta_1(r-1)$ for $r \geqq 5$. Inducing this element in $\widetilde{R}_{n,\mathrm{Top}}^{free}(C(2^{r-1}))$ gives

$$2(\alpha_2(r) + \beta_2(r)) + a_1^{(2^{r-4})}(r) - a_1^{(0)}(r) + a_1^{(2^{r-4}+1)}(r) - a_1^{(1)}(r)$$

and applying part (iii) now gives $2(\alpha_2(r) + \beta_2(r)) = 0$. The required similarities for $s > 2$ are obtained by inducing up from this one, and using the relations $2a_1^{(i)}(r) = 0$ again to remove the lower terms in the formula from part (v). □

The proof of Theorem E. We already have the generators and relations claimed in the statement of Theorem E, so it remains to eliminate all other possible relations. In Example 9.8 we proved that $\widetilde{R}_{\mathrm{Top}}^{free}(C(8)) = \mathbf{Z}/4$ generated by $\alpha_1 = t - t^5$, where $0 \neq 2\alpha_1 \in \widetilde{R}_{n,\mathrm{Top}}^{free}(C(8)$. For $G = C(16)$, we have

$$\widetilde{R}_{\mathrm{Top}}^{free}(C(16)) = \langle \alpha_2, \alpha_1, \beta_1 \rangle$$

and we observe that the elements $2\alpha_2 + \alpha_1$ and $2\alpha_2 + \beta_1$ are not Galois invariant, and hence do not lie in $\widetilde{R}_{n,\mathrm{Top}}^{free}(G)$ by [**II**], Theorem 13.1(iii). Therefore $\widetilde{R}_{\mathrm{Top}}^{free}(C(16)) = \mathbf{Z}/4 \oplus \mathbf{Z}/2 \oplus \mathbf{Z}/2$ as claimed in Theorem E.

We now assume the result for $H = C(2^{r-1})$, with $r \geqq 5$. By applying the inductive assumption, it is not difficult to give generators for the subgroup $\ker \mathrm{Res}_H \cap \widetilde{R}_{h,\mathrm{Top}}^{free}(G)$. Indeed, a generating set consists of the elements (type I)

$$\langle 2^{\ell-1}\alpha_\ell + \alpha_1, \alpha_1 + \beta_1 \mid 2 \leqq \ell \leqq r - 2 \rangle$$

together with the elements (type II)

$$\langle 2^{\ell-2}(\alpha_\ell - \beta_\ell) \mid 3 \leqq \ell \leqq r - 3 \rangle,$$

where the type II elements appear for $r \geqq 6$. Notice that none of the generators (except $\alpha_1 + \beta_1$) are in $\widetilde{R}_{n,\mathrm{Top}}^{free}(G)$, and all the generators have exponent 2.

Let $\gamma_\ell = 2^{\ell-1}\alpha_\ell + \alpha_1$ and consider an linear relation among type I elements of the form

$$\sum n_\ell \gamma_\ell + \epsilon(\alpha_1 + \beta_1) = 0$$

If $\#\{n_\ell \neq 0\}$ is odd, then the left side is not Galois invariant, so it can't be a relation. On the other hand, if $\#\{n_\ell \neq 0\}$ is even, then we can write the first term as a sum of terms $2^{\ell_1-1}\alpha_{\ell_1} - 2^{\ell_2-1}\alpha_{\ell_2} \in \widetilde{R}_h^{free}(G)$ with each $\ell_i \geqq 2$. Suppose first that $\epsilon = 1$. If the first sum on the lefthand side was in $\widetilde{R}^{free}_{n,\mathrm{Top}}(G)$, then its surgery obstruction would be a square and hence trivial by Theorem 9.4. This contradicts the fact that $\alpha_1 + \beta_1 \neq 0$.

We are left with the possibility that $\#\{n_\ell \neq 0\}$ is even and $\epsilon = 0$. Suppose that $\ell_0 \leqq r-3$ is the minimal index such that $n_{\ell_0} = 1$, and write the left side as

$$2^{\ell_0 - 1}\big[\sum_{\ell > \ell_0} n_\ell 2^{\ell-\ell_0}\alpha_\ell + \alpha_{\ell_0}\big] = 0 \ .$$

However, if we call the term in brackets ω and restrict it $(\ell_0 - 1)$ steps to the subgroup of index 2^{ℓ_0-1} in G, we get

$$\sum_{\ell > \ell_0} n_\ell 2^{\ell-\ell_0}\alpha_{\ell-\ell_0+1} + \alpha_1$$

and this is not Galois invariant. We can arrange the signs of the coefficients n_ℓ so that $\omega \in \widetilde{R}^{free}_{h,\mathrm{Top}}(G)$. Since twice this element is trivial, its normal invariant order equals 2. Now by [**II**], Theorem 13.1(iv) we conclude that the normal invariant order (over G) of the bracketed term ω must be 2^{ℓ_0}. Hence there is no relation of this form.

Next we let $\xi_\ell = 2^{\ell-2}(\alpha_\ell - \beta_\ell)$, and suppose that we have a relation of the form

$$\sum_{\ell \geq \ell_0} n_\ell \xi_\ell = 0$$

where ℓ_0 is the minimal non-zero coefficient index as before. If $\#\{n_\ell \neq 0\}$ is odd, then the left-hand side fails the first congruence test for the normal invariant and so it can't be a relation. If $\#\{n_\ell \neq 0\}$ is even we write the left side as $2^{\ell_0-2}\omega$, with $\omega \in \widetilde{R}^{free}_{h,\mathrm{Top}}(G)$, and restrict ω down to a subgroup K of index $2^{\ell_0-2} \leqq 2^{r-5}$. The restriction has the form

$$\mathrm{Res}_K(\omega) = \sum_{\ell > \ell_0} n_\ell 2^{\ell-\ell_0}(\alpha_{\ell-\ell_0+2} - \beta_{\ell-\ell_0+2}) + (\alpha_2 + \beta_2)$$

Since $\#\{n_\ell \neq 0\}$ is even, this can be re-written as a sum $\mathrm{Res}_K(\omega) = \theta - \sigma(\theta)$ with $\theta \in \widetilde{R}^{free}_{h,\mathrm{Top}}(K)$. If $\mathrm{Res}_K(\omega) \in \widetilde{R}^{free}_{n,\mathrm{Top}}(K)$, this would imply that its surgery obstruction was trivial, hence $\mathrm{Res}_K(\omega) = 0$. But restriction one more step down gives $\alpha_1 + \beta_1 \neq 0$ and we have a contradiction. It follows that the normal invariant order of $\mathrm{Res}_K(\omega)$ equals 2, and that of ω is 2^{ℓ_0-1} so the original relation was trivial.

We are left with the possibility of further relations among the type I and type II elements of the form

$$\sum n_\ell \gamma_\ell + \sum m_\ell \xi_\ell + \epsilon(\alpha_1 + \beta_1) = 0.$$

It is easy to reduce to a relation of the form

$$\sum n_\ell 2^{\ell-1}\alpha_\ell + \sum m_\ell 2^{\ell-2}(\alpha_\ell - \beta_\ell) = 0$$

where $\#\{n_\ell \neq 0\} = 2 \cdot \nu$ is even and $\#\{m_\ell \neq 0\} + \nu$ is even. This follows from the Galois invariance and the fact that the p_1–obstruction for normal cobordism is non-trivial for the order 2 elements of the form $(2^{\ell_1-1}\alpha_{\ell_1} - 2^{\ell_2-1}\alpha_{\ell_2})$.

Now if ℓ_0 denotes the minimal index such that n_ℓ or m_ℓ is non–zero, we have two cases. First, if $m_{\ell_0} \neq 0$, we factor out 2^{ℓ_0-2} and restrict our relation to the subgroup K of index

2^{ℓ_0-2}. We obtain

$$2^{\ell_0-2}\big[\sum_{\ell\geqq\ell_0} n_\ell 2^{\ell-\ell_0+1}\alpha_{\ell-\ell_0+2}+\sum_{\ell>\ell_0} m_\ell 2^{\ell-\ell_0}(\alpha_{\ell-\ell_0+2}-\beta_{\ell-\ell_0+2})+(\alpha_2-\beta_2)\big]$$

If the normal invariant obstruction for the term in brackets vanishes, then its surgery obstruction is zero: when ν is even we can rewrite the sum of type II elements as above to get a Reidemeister torsion obstruction of the form $\theta-\sigma(\theta)$, with $\theta\in\widetilde{R}^{free}_{h,\mathrm{Top}}(K)$. The remaining type I terms are collected in pairs $(2^{\ell_1+1}\alpha_{\ell_1+2}-2^{\ell_2+1}\alpha_{\ell_2+2})$ whose surgery obstructions are squares. When ν is odd, we replace $\alpha_2-\beta_2$ by the expression $(\alpha_2-2^t\alpha_{t+2})-(\beta_2-2^t\beta_{t+2})+2^t(\alpha_{t+2}-\beta_{t+2})$ whose torsion has the form $\theta-\sigma(\theta)$ plus a square, and continue as for ν even. The vanishing of this surgery obstruction contradicts $(\alpha_1+\beta_1)\neq 0$ on restricting one step further down.

In the remaining case, if $n_{\ell_0}\neq 0$ and $m_{\ell_0}=0$, we factor out 2^{ℓ_0-1} and restrict to the subgroup of index 2^{ℓ_0-1}. As above, the other factor is not Galois invariant and we get a contradiction to the existence of a normal invariant. This completes the proof. □

11. Non–Linear Similarity for Cyclic 2-Groups

In this final section we will apply our previous results to give explicit necessary and sufficient conditions for the existence of a non-linear similarity $V_1\oplus W\sim_t V_2\oplus W$ for representations of finite cyclic 2-groups. The main result is Theorem 11.6.

Lemma 11.1. *Let $G=C(2^r)$ be a cyclic 2-group. A basis for $\widetilde{R}^{free}_t(C(8))$ is given by $4(t-t^5)$. A basis for $\widetilde{R}^{free}_t(G)$, $r\geqq 4$, is given by the elements*

(i) *$2^s\alpha_s(r)$ for $1\leqq s\leqq r-2$, and*
(ii) *$2(\alpha_1(r)+\beta_1(r))$,*

together with (provided $r\geqq 5$) the elements

(iii) *$2^{s-1}(\alpha_s(r)+\beta_s(r))$ for $2\leqq s\leqq r-3$, and*
(iv) *$\gamma_s^{(i)}(r)=(a_s^{(i)}(r)-a_s^{(i+2)}(r))$ for $0\leqq i<2^{r-s-2}-2$ and $1\leqq s\leqq r-4$.*

Proof. This an immediate consequence of Theorem E. □

The next step is to rewrite the basis is a more convenient form. It will be useful to introduce some notation for certain subsets of $\widetilde{R}^{free}(G)$. Let

$$\mathcal{A}(r)=\{2\alpha_1(r),2(\alpha_1(r)+\beta_1(r))\}$$

for $r\geqq 4$ and set $\mathcal{A}(3)=\{4\alpha_1(3)\}$. Next, let

$$\widehat{\mathcal{B}}(r)=\{2\alpha_1(r),\alpha_1(r)+\beta_1(r)\}$$

for $r\geqq 4$, and let $\widehat{\mathcal{B}}(3)=\{2\alpha_1(3)\}$. Finally, let

$$\mathcal{C}(r)=\{\gamma_s^{(i)}(r)\,|\,0\leqq i<2^{r-s-2}-2 \text{ and } 1\leqq s\leqq r-4\}$$

when $r\geqq 5$, and otherwise $\mathcal{C}(r)=\emptyset$.

Recall the notation G_k for the subgroup of index 2^k in G. We let Ind_k denote induction of representations from G_k to G, and Res_k denote restriction of representations from G_k to G_{k+1}. Define

$$\mathcal{B}(r)=\{\mathrm{Ind}_k(\chi)\,|\,\chi\in\widehat{\mathcal{B}}(r-k)\}$$

for $r \geqq 4$ and $0 < k < r-2$. Note that $x \in \mathcal{B}(r)$ implies that $x = \mathrm{Ind}_k(\chi)$ and $\mathrm{Res}_k(\chi) = 0$.

Lemma 11.2. *The free abelian group $\widetilde{R}_t^{free}(G)$ has an integral basis given by the elements in the set $\mathcal{A}(r) \cup \mathcal{B}(r) \cup \mathcal{C}(r)$.*

Proof. This follows easily from the relations in Lemma 10.1, particularly the induction formula

$$\mathrm{Ind}_H(a_{s-1}^{(i)}(r-1)) = 2a_s^{(i)}(r) - (a_1^{(i)}(r) - a_1^{(2^{r-s-2}+i)}(r))$$

valid for $2 \leqq s \leqq r-2$. Notice that the second term on the right-hand side is just $\alpha_1(r) - \beta_1(r)$ if $s = r-2$, and otherwise it is a linear combination of the basis elements $\gamma_1^{(j)}(r)$. The result is easy for $r = 3$ or $r = 4$, and the inductively we assume it for $r-1$. Then $\mathrm{Ind}_H(2^{s-1}\alpha_{s-1}(r-1)) = 2^s\alpha_s(r)$, plus terms in $\ker \mathrm{Res}_H$ which are all contained in the span of $\mathcal{A}(r)$ and $\mathcal{C}(r)$. Similarly,

$$\mathrm{Ind}_H(2^{s-2}(\alpha_{s-1}(r-1) + \beta_{s-1}(r-1))) = 2^{s-1}(\alpha_s(r) + \beta_s(r)),$$

plus terms in the span of $\mathcal{A}(r)$ and $\mathcal{C}(r)$. This shows that integral linear combinations of the set $\mathcal{A}(r) \cup \mathcal{B}(r) \cup \mathcal{C}(r)$ span $\widetilde{R}_t^{free}(G)$. Since $\mathrm{Res}_H(\gamma_s^{(i)}(r)) = \gamma_{s-1}^{(i)}(r-1)$, $\mathrm{Res}_H \circ \mathrm{Ind}_H = 2$, and $\mathcal{A}(r) \subset \ker \mathrm{Res}_H$, we conclude by induction that there are no non-trivial integral relations among the elements of $\mathcal{A}(r) \cup \mathcal{B}(r) \cup \mathcal{C}(r)$. □

Using the basis given in Lemma 11.2, we now define the set of *weights* $\theta(x) = \{i_1, i_2, \dots, i_\ell\}$ of an element $x \in \widetilde{R}_t^{free}(G)$. This will be a subset of $\{1, 2, \dots, r-2\}$ arranged in strictly ascending order. It will be used to identify the minimal set of isotropy subgroups needed for the construction of a non-linear similarity for $x \in \widetilde{R}_t^{free}(G)$.

Definition 11.3. The weights for $x \in \mathcal{A}(r) \cup \mathcal{B}(r) \cup \mathcal{C}(r)$ are given as follows:

(i) If $x \in \ker \mathrm{Res}_H$ then $\theta(x) = \{1\}$.
(ii) If $x = \mathrm{Ind}_k(2\alpha_1(r-k))$, for $r > r-k \geqq 4$, then $\theta(x) = \{k+1\}$.
(iii) If $x = \mathrm{Ind}_k(\alpha_1(r-k) + \beta_1(r-k))$, for $r > r-k \geq 4$, or $x = \mathrm{Ind}_k(2\alpha_1(3))$, then $\theta(x) = \{k, k+1\}$.
(iv) If $x = \gamma_s^{(i)}(r)$, then $\theta(x) = \{1, 2, \dots, s\}$.

If $x = \sum n_\ell$ is an integral linear combination of elements $\omega_\ell \in \mathcal{A}(r) \cup \mathcal{B}(r) \cup \mathcal{C}(r)$, then $\theta(x) = \bigcup\{\theta(\omega_\ell) \,|\, n_\ell \neq 0\}$.

In other words, after collecting the indices of the subgroups involved in the unique linear combination of basis elements for x, we arrange them in ascending order ignoring repetitions to produce $\theta(x)$.

Definition 11.4. We say that an element $x \in \widetilde{R}_t^{free}(G)$ is *even* if $x = \sum n_\ell \omega_\ell$, $\omega_\ell \in \mathcal{A}(r) \cup \mathcal{B}(r) \cup \mathcal{C}(r)$, has $n_\ell \equiv 0 \bmod 2$ whenever one of the following holds:

(i) $\omega_\ell = 2\alpha_1(r) \in \mathcal{A}(r)$, or
(ii) $\omega_\ell \in \mathcal{C}(r)$.

Otherwise, we say the element x is *odd*. An element x has *mixed* type if x is even, but $n_\ell \not\equiv 0 \bmod 2$ for some $\omega_\ell = \mathrm{Ind}(2\alpha_1(r-k))$ with $r > r-k \geqq 4$. Such an element x has *depth* equal to the minimum k for which x contains a constituent $\mathrm{Ind}(2\alpha_1(r-k))$ with odd multiplicity.

Lemma 11.5. *An element* $x = [V_1 - V_2] \in \widetilde{R}_t^{free}(G)$ *is even if and only if its Reidemeister torsion invariant* $\{\Delta(V_1)/\Delta(V_2)\} = 0$ *in* $H^1(\mathrm{Wh}(\mathbf{Z}G^-)/\mathrm{Wh}(\mathbf{Z}H))$.

Proof. The Reidemeister torsion invariants for elements of $\ker \mathrm{Res}_H$ lie in $H^1(\mathrm{Wh}(\mathbf{Z}G^-))$, which has a basis of units $U_{1,i}$ for $1 \leqq i < 2^{r-1}$ and $i \equiv 1 \bmod 4$ (see [5, §5]). In particular, $\{\Delta(2\alpha_1(r))\} = U_{1,1} \neq 0$, and $\{\Delta(\alpha_1(r) + \beta_1(r))\} = U_{1,5} \neq 0$, but $\{\Delta(2(\alpha_1(r) + \beta_1(r)))\} = U_{1,5}^2 = 0$ in $H^1(\mathrm{Wh}(\mathbf{Z}G^-))$. Note that $H^1(\mathrm{Wh}(\mathbf{Z}G^-))$ injects into $H^1(\mathrm{Wh}(\mathbf{Z}G^-)/\mathrm{Wh}(\mathbf{Z}H))$.

Suppose first that $x = [V_1 - V_2] \in \widetilde{R}_t^{free}(G)$ is even. It follows that its Reidemeister torsion invariant is either a square or induced up from H, so $\{\Delta(V_1)/\Delta(V_2)\} = 0$ in $H^1(\mathrm{Wh}(\mathbf{Z}G^-)/\mathrm{Wh}(\mathbf{Z}H))$. Conversely, suppose that $x \in \widetilde{R}_t^{free}(G)$ has an odd coefficient n_ℓ for $\omega_\ell = 2\alpha_1(r)$ or $\omega_\ell = \gamma_1^{(i)}(r)$ in $\ker \mathrm{Res}_H$. In these cases, $\{\Delta(V_1)/\Delta(V_2)\} \neq 0$ in $H^1(\mathrm{Wh}(\mathbf{Z}G^-)/\mathrm{Wh}(\mathbf{Z}H))$.

Finally, suppose that n_ℓ is odd for some $\omega_\ell = \gamma_s^{(i)}(r) \in \mathcal{C}(r)$ with $s \geqq 2$. If the Reidemeister torsion invariant for ω_ℓ were trivial in $H^1(\mathrm{Wh}(\mathbf{Z}G^-)/\mathrm{Wh}(\mathbf{Z}H))$, then its image under the twisted restriction map would also be trivial. This is the map defined by composing the twisting isomorphism

$$H^1(\mathrm{Wh}(\mathbf{Z}G^-)/\mathrm{Wh}(\mathbf{Z}H)) \cong H^0(\mathrm{Wh}(\mathbf{Z}G)/\mathrm{Wh}(\mathbf{Z}H))$$

with Res_H, followed by another twisting isomorphism at the index two level. But

$$\mathrm{Res}_H(\gamma_s^{(i)}(r)) = \gamma_{s-1}^{(i)}(r-1) ,$$

so after restricting $s-1$ steps we arrive at $\gamma_1^{(i)}(r-s+1) \in \ker \mathrm{Res}_{s-1}$, and a contradiction as in the previous case. □

For $2 \leqq i \leqq r-2$, let W_i denote any irreducible 2-dimensional real representation of G with isotropy group G_i, let $W_1 = \mathbf{R}_-$, and let $W_0 = \mathbf{R}_+$. The terms defined in Definitions 11.3 and 11.4 will be used in the statement of our classification result.

Theorem 11.6. *Let* $G = C(2^r)$ *and* V_1, V_2 *be free* G*-representations. Suppose that* $x = [V_1 - V_2] \in \widetilde{R}_t^{free}(G)$. *Then* $V_1 \oplus W \sim_t V_2 \oplus W$ *for a given* G*-representation* W *if and only if the representation* W *contains:*

(i) *a summand* W_k *for each* $k \in \theta(x)$,
(ii) *a summand* W_t *for some* $0 \leqq t \leqq depth(x)$ *when* x *has mixed type, and*
(iii) *a summand* $\mathbf{R}_+$ *when* x *is odd.*

Proof. The sufficiency of the given conditions follows immediately from Theorem 9.4, the basic list of 6-dimensional similarities in [8, Thm.1 (iii)], and the following commutative diagram

$$\begin{array}{ccc} L^h_{2k}(\mathbf{Z}H) & \xrightarrow{\quad i_* \quad} & L^h_{2k}(\mathbf{Z}G) \\ \downarrow{\scriptstyle trf_{\mathrm{Res}_H W}} & & \downarrow{\scriptstyle trf_W} \\ L^h_{2k+m}(\mathcal{C}_{\mathrm{Res}\, W,H}(\mathbf{Z})) & \xrightarrow{i_*} & L^h_{2k+m}(\mathcal{C}_{W,G}(\mathbf{Z})) \end{array}$$

or the corresponding L^p version if W contains an $\mathbf{R}_+$ sub-representation. The summand $\mathbf{R}_+$ will be unnecessary exactly when $\{\Delta(V_1)/\Delta(V_2)\} = 0$ in $H^1(\mathrm{Wh}(\mathbf{Z}G^-)/\mathrm{Wh}(\mathbf{Z}H))$, by Theorem 9.2. By Lemma 11.5 this happens precisely when x is even.

For all elements $[V_1 - V_2] \in \widetilde{R}_t^{free}(H)$ we have $\dim V_i = 2k \equiv 0 \bmod 4$. To handle the surgery obstructions of induced representations, note that

$$\{\Delta(\operatorname{Ind} V_1)/\Delta(\operatorname{Ind} V_2)\} = i_*\{\Delta(V_1)/\Delta(V_2)\} \in H^0(\mathrm{Wh}(\mathbf{Z}G))$$

for any $x = [V_1 - V_2] \in \widetilde{R}_t^{free}(H)$. For the surgery obstruction of an H-homotopy equivalence $f\colon S(V_2) \to S(V_1)$ we have the relation

$$\sigma(\operatorname{Ind}(f)) = i_*(\sigma(f)) + i_*(\beta) \in L_0^h(\mathbf{Z}G)$$

for some $\beta \in L_0^s(\mathbf{Z}H)$. It follows that

$$i_*(trf_{\operatorname{Res} W}(\sigma(f))) = trf_W(\sigma(\operatorname{Ind}(f))) \in L_m^h(\mathcal{C}_{W,G}(\mathbf{Z}))$$

provided that $\operatorname{Res} W$ contains an $\mathbf{R}_-$ summand. This formula gives the existence of unstable similarities for elements $\operatorname{Ind}_k(\chi) \in \mathcal{B}(r)$ under the given conditions on W.

The necessity of condition (i) follows immediately from [17, Thm. 5.1], once we relate the odd p-local components used there to our setting. The surgery obstruction group $L_{2k}^h(\mathbf{Z}G)$ has a natural splitting, after localizing at any odd prime, indexed by the divisors of $|G| = 2^r$. An element $\sigma \in L_{2k}^h(\mathbf{Z}G)$ of infinite order will have a non-trivial projection in the "top" 2^r-component provided that $\operatorname{Res}_H(\sigma) = 0$. This applies to our basis elements $2\alpha_1(r)$, $2(\alpha_1(r) + \beta_1(r))$, $4\alpha_1(3)$, and $\gamma_1^{(i)}(r)$. If any of these occurs as a constituent in x, a summand $\mathbf{R}_-$ is necessary to produce the non-linear similarity.

More generally, σ will have a non-trivial projection in the 2^{r-k}-component for $k > 0$ provided that $\operatorname{Res}_{G_k}(\sigma)$ is non-zero but $\operatorname{Res}_{G_{k+1}}(\sigma) = 0$. In this case the representation W must contain at least one summand W_{k+1} which restricts to $\mathbf{R}_-$ for the subgroup G_k. To see this we restrict the surgery obstruction for our element $x = [V_1 - V_2]$ to each G_k for $k+1 \in \theta(x)$ and apply [17, Thm. 5.1]. This establishes part of condition (i) for x containing any one of the basis elements $\operatorname{Ind}_k(\chi) \in \mathcal{B}(r)$. We will deal below with the necessity of the additional summand W_k for the basis elements $\operatorname{Ind}_k(\alpha_1(r-k) + \beta_1(r-k))$, or $\operatorname{Ind}_k(2\alpha_1(3))$, having $\theta = \{k, k+1\}$.

For the elements $\gamma_s^{(i)}(r)$ with $s \geqq 2$, we use the relation $(2x - \operatorname{Ind}_H(\operatorname{Res}_H x)) \in \ker \operatorname{Res}_H$ and the induction formula in Lemma 10.1 again. This shows that the elements $\gamma_s^{(i)}(r)$ have a non-trivial projection into each component of index $\leqq 2^s$, and we again apply [17, Thm. 5.1]. Condition (i) is now established for any x containing some $\gamma_s^{(i)}(r)$ as a constituent.

Condition (ii) applies only to elements x of mixed type. These contain a constituent $\operatorname{Ind}_k(2\alpha_1(r-k))$, for $r - k \geqq 4$, with odd multiplicity. It says that W must contain a summand which restricts to a sum of $\mathbf{R}_+$ representations over the subgroup G_k. Suppose if possible that $W = W_{k+1} \oplus U$ where U has isotropy groups contained in G_{k+1}. Then $K_1(\mathcal{C}_{W_{k+1},G}(\mathbf{Z})) \cong K_1(\mathcal{C}_{W,G}(\mathbf{Z}))$ by [**II**], Lemma 6.1, and

$$\mathrm{Wh}(\mathcal{C}_{W_{k+1},G}(\mathbf{Z})) \cong \mathrm{Wh}(\mathbf{Z}G)/\,\mathrm{Wh}(\mathbf{Z}G_{k+1})\ .$$

However, the results of [5, Prop. 1.2] show that the inclusion induces an injection

$$H^0(\mathrm{Wh}(\mathbf{Z}G_k)/\,\mathrm{Wh}(\mathbf{Z}G_{k+1})) \to H^0(\mathrm{Wh}(\mathbf{Z}G)/\,\mathrm{Wh}(\mathbf{Z}G_{k+1}))\ .$$

It follows that the surgery obstruction $trf_W(\sigma)$ is non-zero, contradicting the existence of such a non-linear similarity.

To see that a summand W_k is also necessary when x has a constituent of the form $\mathrm{Ind}_k(\alpha_1(r-k)+\beta_1(r-k))$, $r \geqq 4$, or $\mathrm{Ind}_k(2\alpha_1(3))$, we start with the argument of the last paragraph again. It shows that W must contain some W_t for $t \leqq k$. However the representations W_t with $t < k$ all restrict to $\mathbf{R}_+$ or $\mathbf{R}_+^2$ over G_k, so it is enough to eliminate similarities of the form $W = W_2 \oplus \mathbf{R}_+$ for G with $k = 1$. Restriction from G to G_{k-1} then gives the general case. However, a non-linear similarity of this form is ruled out by considering the surgery obstruction $trf_W(\mathrm{Ind}_H(\sigma)) \in L_2^p(\mathcal{C}_{W,G}(\mathbf{Z}))$. Over the subgroup H the representation $\mathrm{Res}_H W_2 = \mathbf{R}_-^2$, and the surgery obstruction $trf_{\mathrm{Res}_H W}(\sigma)$ is computed from the twisting diagrams tabulated in [18, Appendix 2]. In particular, it is non-zero in $L_2^p(\mathbf{Z}H)/L_2^p(\mathbf{Z}K)$ (see [18, Table 2: $U \to U$, p.123]), which injects into $L_2^p(\mathcal{C}_{\mathrm{Res}_H W,H}(\mathbf{Z}))$, where K denotes the subgroup of index 4 in G. Since

$$0 \to L_2^p(\mathbf{Z}G)/L_2^p(\mathbf{Z}K) \to L_2^p(\mathcal{C}_{W,G}(\mathbf{Z})) \to L_0^p(\mathbf{Z}K) \to 0$$

we check that the inclusion $H \subset G$ induces an injection

$$L_2^p(\mathcal{C}_{\mathrm{Res}_H W,H}(\mathbf{Z})) \to L_2^p(\mathcal{C}_{W,G}(\mathbf{Z}))$$

and hence the surgery obstruction $trf_W(\sigma) \neq 0$ from the commutative diagram above. □

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