First, some questions for clarification:

1. If $\beta \cdot s \le \gamma$, for example, then $s \le \beta \uparrow \gamma$ by Axiom (3); so does this mean that $s = \beta \uparrow \gamma$ since *s* is the upper bound of *P*? If so, then I'm not sure if I understand this yet in the context of subgroup properties.
2. What exactly is Δ (at least in the context of subgroup properties), and would (could) it serve as an identity for the / operation?
3. Would you agree that, in the context of subgroup properties, we can say
   (a) α is transitive iff $\alpha \cdot \alpha \le \alpha$, and
   (b) α is persistent iff $\alpha / s \le \alpha$.

I spent some time this weekend writing out proofs and generalizing ideas with some success.

Restatement of Lemma 5.

For any $\alpha, \beta, \gamma \in P$,

(a) $(\alpha \uparrow \beta) \cdot (\beta \uparrow \gamma) \le \alpha \uparrow \gamma$
(b) $(\alpha \downarrow \beta) \cdot (\beta \downarrow \gamma) \le \alpha \downarrow \gamma$
(c) $(\alpha \uparrow \beta) / (\gamma \downarrow \beta) \le \alpha \uparrow \gamma$
(d) $(\alpha \downarrow \beta) / (\gamma \uparrow \beta) \le \alpha \downarrow \gamma$

Corollary 6 lists some special cases from Lemma 5 with $\gamma = \alpha$ or β.

I was excited to discover that Lemma 7 can be completely generalized. (The results are not in the same order as in my talk slides.)

Restatement of Lemma 7.

For any $\alpha, \beta, \gamma, \sigma \in P$,

(a) $\beta \cdot \sigma \le \gamma \;\Rightarrow\; (\alpha \uparrow \beta) \cdot \sigma \le \alpha \uparrow \gamma$
(b) $\gamma / \sigma \le \beta \;\Rightarrow\; (\alpha \downarrow \beta) \cdot \sigma \le \alpha \downarrow \gamma$
(c) $\beta / \sigma \le \gamma \;\Rightarrow\; (\alpha \uparrow \beta) / \sigma \le \alpha \uparrow \gamma$
(d) $\gamma \cdot \sigma \le \beta \;\Rightarrow\; (\alpha \downarrow \beta) / \sigma \le \alpha \downarrow \gamma$

But I realized that even Lemma 7 is a corollary to Lemma 5. For instance, if $\beta \cdot \sigma \le \gamma$, then $\sigma \le \beta \uparrow \gamma \le (\alpha \uparrow \beta) \uparrow (\alpha \uparrow \gamma)$ by Axiom (3) and Lemma 5(a); this is Lemma 7(a). So the good news is that the 'cancellation laws' seem to have deeper implications; the bad news is that the list of results is shortened by four.

We can generalize Lemma 9.

Restatement of Lemma 9.
For any $\alpha, \beta, \gamma \in P$,

(a) $\beta \le \gamma \;\Rightarrow\; \alpha \uparrow \beta \le \alpha \uparrow \gamma$ ($\uparrow$ is isotone in the second variable)
(b) $\beta \le \gamma \;\Rightarrow\; \beta \downarrow \alpha \le \gamma \downarrow \alpha$ ($\downarrow$ is isotone in the first variable)
(c) $\gamma \le \beta \;\Rightarrow\; \beta \uparrow \alpha \le \gamma \uparrow \alpha$ ($\uparrow$ is antitone in the first variable)
(d) $\gamma \le \beta \;\Rightarrow\; \alpha \downarrow \beta \le \alpha \downarrow \gamma$ ($\downarrow$ is antitone in the second variable)

I was determined to prove Lemma 8 without additional axioms, and had no success, but I am not yet convinced one way or the other. I tentatively restate it here with some, possibly faulty, interpretation. I also did not investigate whether Lemma 8 can be generalized.

Interpretation of Lemma 8.
For any $\alpha, \beta \in P$,

(a) $$\left.\begin{array}{l} \alpha / s \le \alpha \\ \beta / s \le \beta \end{array}\right\} \Rightarrow \alpha \uparrow \beta \le \beta \downarrow \alpha$$

(b) $$\left.\begin{array}{l} \alpha \cdot \alpha \le \alpha \\ \beta \cdot \alpha \le \alpha \\ \beta \cdot \beta \le \alpha \\ \beta \cdot \beta \le \beta \end{array}\right\} \Rightarrow \alpha \downarrow \beta \le \beta \uparrow \alpha$$

I am wondering what types of questions we could ask here from an algebraic and/or lattice-theoretic standpoint. Initially, it is interesting to look at the relationships between the four binary operations, but I assume that there are other elementary questions.