Given the equation $x^2y'' - xy' + y = 0$ with solution $y_1 = -x$; verify that y_1 is solution, find a second independent solution y_2, and check your answer.

First, we verify that $y_1 = -x$ is a solution. To do this, we compute y_1' and y_1'', and plug them in to the original equation to make sure it simplifies to 0.

$$y_1 = -x \Rightarrow y_1' = -1 \Rightarrow y_1'' = 0$$

Plugging in gives $x^2(0) - x(-1) + (-x) = x - x = 0$. So we've verified that y_1 is in fact a solution.

Now to find a second solution, y_2. The trick we always use here is to let $y_2 = uy_1 = -ux$ for some unknown function u of x. The work we are about to do is to figure exactly what function of x u must be in order to ensure that y_2 is a solution.
We want our function to look like the general form of a second order linear differential equation. The general form is:

$$y'' + p(x)y' + q(x)y = 0.$$

For our equation, we just divide through by x^2:

$$y'' - x^{-1}y' + x^{-2}y = 0.$$

Now, our ultimate goal is that y_2 be a solution to this, i.e., we want $y_2'' - x^{-1}y_2' + x^{-2}y_2 = 0$. We can replace y_2 in this expression with uy_1, since that's how y_2 was defined. So we have

$$(-ux)'' - x^{-1}(-ux)' + x^{-2}(-ux) = 0.$$

By the product rule,

$$(-ux)' = -u - u'x$$

and

$$\begin{aligned}(-ux)'' &= -u' - (u' + u''x) \\ &= -u''x - 2u'\end{aligned}$$

Thus, our equation above becomes

$$\begin{aligned} -u''x - 2u' - x^{-1}(-u - u'x) + x^{-2}(-ux) &= 0 \\ -u''x - 2u' + ux^{-1} + u' - ux^{-1} &= 0 \\ -u''x - u' &= 0 \end{aligned}$$

So we just need to find a u satisfying this equation. This is a second order differential equation, but if we let $v = u'$, we have just

$$\begin{aligned} -v'x - v &= 0 \\ -\frac{dv}{dx}x &= v \\ \int \frac{dv}{v} &= -\int \frac{dx}{x} \\ \ln(v) &= -\ln(x) \\ \ln(v) &= \ln(x^{-1}) \\ v &= x^{-1} \end{aligned}$$

So $u' = x^{-1}$, which means $u = \ln(x)$. Notice in these two integrals, we're ignoring the constants of integration. We can do this because we just need to find any u that makes the whole thing work. The easiest thing to do therefore is let the constants be zero.

Since $y_2 = uy_1$, we have $y_2 = -x\ln(x)$, making the final solution $y = -c_1x - c_2x\ln(x)$, or just $y = c_1x + c_2x\ln(x)$, since c_1 and c_2 are arbitrary constants.

To check that our solution works, we need only verify that y_2 works. If y_2 works, then since we know y_1 works and we know that differentiation is a linear operator, we automatically know that $c_1y_1 + c_2y_2$ will work.

Check:

$$\begin{aligned} y_2 &= x\ln(x) \\ y_2' &= \ln(x) + 1 \\ y_2'' &= x^{-1} \end{aligned}$$

$$x^2(x^{-1}) - x(\ln(x) + 1) + x\ln(x) = x - x\ln(x) - x + x\ln(x) = 0$$

So, our solution is verified.