MATH 371 - Summer 2007 Exam 1 Name:________________________

Five questions, 20 points each. Show all work for full credit.

1. Solve the following separable initial value problem.

$$\frac{1}{2}y' = xy - x + y - 1, \quad y(0) = 2$$

*Solution.*

$$\begin{aligned} y' &= 2(x+1)(y-1) \\ \int \frac{dy}{y-1} &= 2\int (x+1)dx \\ \ln|y-1| &= x^2 + 2x + c_1 \\ (y-1) &= ce^{(x^2+2x)} \\ \\ (2-1) &= ce^0 = c \\ c &= 1 \\ \\ y &= e^{(x^2+2x)} + 1 \end{aligned}$$

□

2. Consider the matrix $A = \begin{bmatrix} 2 & 5 \\ 4 & 1 \end{bmatrix}$.

(a) Compute the determinant of $A$.

(b) Is $A$ invertible? Briefly, why or why not?

(c) Find the eigenvalues for $A$. For each, find a corresponding eigenvector.

*Solution.*

(a) $|A| = 2 - 20 = -18$.

(b) $A$ is invertible, since its determinant is nonzero.

(c) $(2-\lambda)(1-\lambda) - 20 = (2 - 3\lambda + \lambda^2) - 20 = \lambda^2 - 3\lambda - 18$.

$\lambda_1 = 6: \begin{array}{r} -4x_1 + 5x_2 = 0 \\ 4x_1 + -5x_2 = 0 \end{array}$. So $\begin{bmatrix} 1 \\ \frac{4}{5} \end{bmatrix}$ is an eigenvector for the eigenvalue 6.

$\lambda_2 = -3: \begin{array}{r} 5x_1 + 5x_2 = 0 \\ 4x_1 + 4x_2 = 0 \end{array}$. So $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector for the eigenvalue $-3$.

□

3. Solve the differential equation $6xydx + (4y + 9x^2)dy = 0$

*Solution.* Let $P(x,y) = 6xy$ and $Q(x,y) = (4y + 9x^2) = 0$. Then $\frac{\partial P}{\partial y} = 6x$ and $\frac{\partial Q}{\partial x} = 18x$. So this equation isn't exact.

$$\frac{1}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = \frac{12x}{6xy} = \frac{2}{y}$$

Since this is a function of $y$, we can multiply through by:

$$e^{2\int y^{-1}dy} = e^{2\ln y} = y^2$$

obtaining

$$6xy^3dx + (4y^3 + 9x^2y^2dy = 0$$

which is exact and has solution $y^4 + 3x^2y^3 = c$. □

4. Let $u(x,y) = 3ye^x$, $y(t) = \ln(t)$, and $x(t) = t^2$. Find the derivative of $u$ with respect to $t$ using both the chain rule and the method of substitution.

*Solution.* Chain rule:

$$\begin{aligned}[3ye^x + 3e^x]\begin{bmatrix} 2t \\ \frac{1}{t} \end{bmatrix} &= 6yte^x + \frac{3e^x}{t} \\ &= e^{t^2}(6t\ln(t) + \frac{3}{t})\end{aligned}$$

Substitution:

$$\begin{aligned} u(t) &= 3\ln(t)e^{t^2} \\ u'(t) &= \frac{3e^{t^2}}{t} + 3\ln(t)(2t)e^{t^2} \\ &= e^{t^2}(\frac{3}{t} + 6t\ln(t))\end{aligned}$$

□

5. Using Picard's Method, find $y_1$ and $y_2$ for the equation

$$y' = x^2 + x + y, \quad y(0) = 1$$

*Solution.*

$$\begin{aligned} y_1 &= 1 + \int_0^x (t^2 + t + 1)dt \\ &= 1 + \frac{x^3}{3} + \frac{x^2}{2} + t \end{aligned}$$

$$\begin{aligned} y_2 &= 1 + \int_0^x (t^2 + t + (1 + t + \frac{t^2}{2} + \frac{t^3}{3}))dt \\ &= 1 + \int_0^x (1 + 2t + \frac{3t^2}{2} + \frac{t^3}{3})dt \\ &= 1 + x + x^2 + \frac{x^3}{2} + \frac{x^4}{12} \end{aligned}$$

□