MATH 371 - Summer 2007 Exam 4 Name: Key

Five questions, 20 points each. Show all work for full credit.

1. Solve the following.

   (a) $x^2y'' + xy' - 3y = 0$

   (b) $x^2y'' - xy' + y = 0$.

(a) $x^2y'' + xy' - 3y = 0$

$a = +1 \quad b = -3$

$m^2 + (1-1)m - 3 = 0$

$m^2 - 3 = 0$

$m = \pm\sqrt{3}$ (2 real roots)

$$y = c_1 x^{\sqrt{3}} + c_2 x^{-\sqrt{3}}$$

(b) $x^2y'' - xy' + y = 0$

$a = -1 \quad b = 1$

$m^2 - 2m + 1 = 0$

$(m-1)^2 = 0$

$m = 1$, (double root)

$$y = c_1 x + c_2 x \ln x$$

$x,\ 1,\ 0: \quad 0 - x + x = 0$ ✓

$x\ln x,\quad \ln x + 1,\quad \frac{1}{x}$

$x - x\ln x - x + x\ln x = 0$ ✓

2. (a) For the following matrix, find both eigenvalues, and a corresponding eigenvector for each.

$$A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$$

(b) Verify that the eigenvectors and eigenvalues you found are correct.

(c) Find the general solution for the system of equations

$$\begin{aligned} y_1' &= 4y_1 + y_2 \\ y_2' &= 2y_1 + 3y_2 \end{aligned}$$

(d) Given initial values $y_1(0) = 0$, and $y_2(0) = 3$, find the particular solution for the system of equations in the previous part.

(a) $\begin{vmatrix} 4-\lambda & 1 \\ 2 & 3-\lambda \end{vmatrix} = (4-\lambda)(3-\lambda) - 2 = 12 - 3\lambda - 4\lambda + \lambda^2 - 2$

$= \lambda^2 - 7\lambda + 10 = (\lambda - 5)(\lambda - 2) = 0$

$\lambda_1 = 5 \quad \lambda_2 = 2$

$\lambda_1 = 5: \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix} \quad -y_1 + y_2 = 0 \quad y_1 = y_2 \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2 = 2: \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \quad 2y_1 + y_2 = 0 \quad y_2 = -2y_1 \quad \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

eigenvalues & eigenvectors: $\left(5, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$ and $\left(2, \begin{bmatrix} 1 \\ -2 \end{bmatrix}\right)$

(b) $\begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 4-2 \\ 2-6 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ ✓

(c) $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = c_1 e^{5x}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{2x}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ or $\begin{aligned} y_1 &= c_1 e^{5x} + c_2 e^{2x} \\ y_2 &= c_1 e^{5x} - 2c_2 e^{2x} \end{aligned}$

(d) $y_1(0) = 0: \quad 0 = c_1 + c_2$

$y_2(0) = 3: \quad 3 = c_1 - 2c_2$

$3 = -c_2 - 2c_2 = -3c_2$

$c_2 = -1 \quad c_1 = 1$

solution: $y_1 = e^{5x} - e^{2x}$

$y_2 = e^{5x} + 2e^{2x}$

3. Solve the following differential equation using the method of undetermined coefficients:

$$y'' - 4y' + 4y = 2\sin(x)$$

y_h:

$y'' - 4y' + 4y = 0$

$\lambda^2 - 4\lambda + 4 = 0$

$(\lambda - 2)^2$

$\lambda = 2$, (double)

$y_h = c_1 e^{2x} + c_2 x e^{2x}$

$r(x) = 2\sin(x)$

y_p (guess): $c_3 \sin(x) + c_4 \cos(x)$

y_p': $c_3 \cos(x) - c_4 \sin(x)$

$-c_3 \sin(x) - c_4 \cos(x)$

$y_p'' - 4y_p' + 4y_p = 4c_3 \sin(x) + 4c_4 \cos(x)$
$+ 4c_4 \sin(x) - 4c_3 \cos(x)$
$- c_3 \sin(x) - c_4 \cos(x)$

$= (3c_3 + 4c_4)\sin(x) + (3c_4 - 4c_3)\cos(x)$ set $= r(x)$ $= 2\sin(x) + 0\cos(x)$

so $3c_3 + 4c_4 = 2$

$3c_4 - 4c_3 = 0$

$c_4 = \frac{4}{3} c_3$

$3c_3 + \frac{16}{3} c_3 = 2$

$\frac{25}{3} c_3 = 2$

$c_3 = \frac{6}{25}$

$c_4 = \frac{4 \cdot 6}{3 \cdot 25} = \frac{8}{25}$

$y_p = \frac{6}{25}\sin(x) + \frac{8}{25}\cos(x)$

$$\boxed{y = c_1 e^{2x} + c_2 x e^{2x} + \frac{6}{25}\sin(x) + \frac{8}{25}\cos(x)}$$

4. Determine the Laplace transforms for the following:

(a) te^{-t}
(b) $\cos^2(t) + e^{2t}$
(c) $\sin(2t)$
(d) $t^3 - 3\delta(t-5)$.

$\mathcal{L}(e^{at}f) = F(s-a)$

$\mathcal{L}(\delta(t-a)) = e^{-as}$

$\cos^2(\theta) = \cos(2\theta) + \sin^2\theta = \cos(2\theta) + 1 - \cos^2\theta = \frac{\cos(2\theta) + 1}{2}$

(a) $a = -1$ $\quad \mathcal{L}(t) = \frac{1}{s^2}$ $\quad \boxed{\mathcal{L}(te^{-t}) = \frac{1}{(s+1)^2}}$

(b) $\cos^2(t) = \frac{1}{2}\cos(2t) + \frac{1}{2}$

$\mathcal{L}(\cos^2(t) + e^{2t}) =$

$\frac{1}{2}\mathcal{L}(\cos(2t)) + \frac{1}{2}\mathcal{L}(1) + \mathcal{L}(e^{2t})$

$$\boxed{= \frac{s}{2(s^2+4)} + \frac{1}{2s} + \frac{1}{s-2}}$$

(c) $\mathcal{L}(\sin(2t)) = \boxed{\frac{2}{s^2+4}}$

(d) $\mathcal{L}(t^3 - 3\delta(t-5)) = \mathcal{L}(t^3) - 3\mathcal{L}(\delta(t-5))$

$= \frac{6}{s^4} - 3e^{-5s}$

5. Solve the equation $y''+2x^3y'-3xy=0$ using power series. Find at least the first 8 coefficients (i.e., a_0 through a_7), and write the general solution as a linear combination of two particular solutions. (I.e., the form of the answer should be as in the examples in class.)

	x^0	x^1	x^2	x^3	x^4	x^5
$-3xy$:	0	$-3a_0$	$-3a_1$	$-3a_2$	$-3a_3$	$-3a_4$
$+2x^2y'$:	0	0	$2a_1$	$4a_2$	$6a_3$	$8a_4$
y'':	$2a_2$	$6a_3$	$12a_4$	$20a_5$	$30a_6$	$42a_7$

$a_2=0$ $\quad a_3=\frac{a_0}{2}$ $\quad a_4=\frac{a_1}{12}$ $\quad a_6=\frac{-a_3}{10}=\frac{-a_0}{20}$ $\quad a_7=\frac{-5a_4}{42}=\frac{-5a_1}{42\cdot 12}$

$a_5=0$

$(42\cdot12=420+84=504)$

$=\frac{-5}{504}a_1$

$$y=a_0\left(1+\frac{x^3}{2}-\frac{x^6}{20}+\dots\right)$$

$$+a_1\left(x+\frac{x^4}{12}-\frac{5}{504}x^7+\dots\right)$$