1. $y' - 3y = e^{3t}$  $y(0)=1$

$h = \int p\,dt$  $p=-3$

$= -3\int dt = -3t$

$e^h = e^{-3t}$  $e^{-3t}y' - 3e^{-3t}y = 1$

$(e^{-3t}y)' = 1$

$e^{-3t}y = t + C$

$y = te^{3t} + Ce^{3t}$

$1 = C$

$y = (t+1)e^{3t}$

Laplace:

$\mathcal{L}(y') = s\mathcal{L}(y) - 1$

$s\mathcal{L}(y) - 1 - 3\mathcal{L}(y) = \mathcal{L}(e^{3t})$

$(s-3)\mathcal{L}(y) - 1 = \frac{1}{(s-3)}$

$(s-3)\mathcal{L}(y) = \frac{1}{(s-3)} + 1$

$\mathcal{L}(y) = \frac{1}{(s-3)^2} + \frac{1}{(s-3)}$

$y = e^{3t}t + e^{3t} = (t+1)e^{3t}$

2. $y' + 4y = e^{-4t}$, $y(0) = 2$

$e^{4t}y' + 4e^{4t}y = 1$

$(e^{4t}y)' = 1$, $e^{4t}y = t + c$, $y = te^{-4t} + ce^{-4t}$

$y(0) = 2$  $2 = ce^{-40}$  $c = 2$

$\boxed{y = te^{-4t} + 2e^{-4t}}$

Laplace:

$\mathcal{L}(y') = s\mathcal{L}(y) - 2$

$s\mathcal{L}(y) - 2 + 4\mathcal{L}(y) = \frac{1}{s+4}$

$(s+4)\mathcal{L}(y) = \frac{1}{s+4} + 2$

$\mathcal{L}(y) = \frac{1}{(s+4)^2} + \frac{2}{(s+4)}$

$\boxed{y = te^{-4t} + 2e^{-4t}}$

3. $y'' - 6y' + 9y = t$ $\quad y(0)=0$, $y'(0)=1$

$y_h$: $\lambda^2 - 6\lambda + 9 = 0$

$(\lambda - 3)^2 = 0$

$y_h = c_1 e^{3t} + c_2 t e^{3t}$

$y_p$: (undetermined coefficients)

guess $c_3 t + c_4 = y_p$

$c_3 = y_p'$

$0 = y_p''$

$y_p'' - 6y_p' + 9y_p \quad | \; r(t)$

$= -6c_3 + 9c_3 t + 9c_4 = t$

$9c_3 = 1 \quad c_3 = \frac{1}{9}$

$-6c_3 + 9c_4 = 0$

$c_4 = \frac{2}{3} c_3 = \frac{2}{27}$

so $y_p = \frac{t}{9} + \frac{2}{27}$

Check: $-6\left(\frac{1}{9}\right) + \frac{9}{9}t + \frac{18}{27}$

$= -\frac{2}{3} + t + \frac{2}{3} = t$ ✓

$y(t) = c_1 e^{3t} + c_2 t e^{3t} + \frac{t}{9} + \frac{2}{27}$

$y(0) = 0 \quad 0 = c_1 + \frac{2}{27} \quad c_1 = -\frac{2}{27}$

$y'(0) = 1 \quad 3c_1 e^{3t} + c_2 e^{3t} + 3c_2 t e^{3t} + \frac{1}{9} = y'$

$y'(0) = 3c_1 + c_2 + \frac{1}{9} = 1$

$-\frac{6}{27} + c_2 + \frac{1}{9} = 1$

$-\frac{2}{9} + c_2 + \frac{1}{9} = 1$

$-\frac{1}{9} + c_2 = 1$

$c_2 = \frac{10}{9}$

$y = -\frac{2}{27} e^{3t} + \frac{10 t e^{3t}}{9} + \frac{t}{9} + \frac{2}{27}$

Laplace $\quad \mathcal{L}(y') = s\mathcal{L}(y)$

$\mathcal{L}(y'') = s^2 \mathcal{L}(y) - 1$

$s^2 \mathcal{L}(y) - 1 - 6s\mathcal{L}(y) + 9\mathcal{L}(y) = \frac{1}{s^2}$

$(s^2 - 6s + 9)\mathcal{L}(y) = \frac{1}{s^2} + 1$

$\mathcal{L}(y) = \frac{1}{s^2(s-3)^2} + \frac{1}{(s-3)^2} \qquad \left(\mathcal{L}^{-1}\left(\frac{1}{(s-3)^2}\right) = e^{3t} t\right)$

Thm 5. $\mathcal{L}\left(\int_0^t f\right) = \frac{\mathcal{L}(f)}{s}$

$\frac{\mathcal{L}(e^{3t} t)}{s} = \frac{1}{s(s-3)^2} = \mathcal{L}\left(\int_0^t x e^{3x} dx\right)$

$\int_0^t x e^{3x} dx = \frac{1}{3} x e^{3x} - \frac{1}{3}\int e^{3x} dx = \frac{1}{3} x e^{3x} - \frac{1}{9} e^{3x} \Big|_0^t$

$u = t \quad du = dt$

$dv = e^{3t} dt \quad v = \frac{1}{3} e^{3t}$

$= \frac{1}{3} t e^{3t} - \frac{1}{9} e^{3t} + \frac{1}{9}$

$\frac{\mathcal{L}(e^{3t} t)}{s^2} = \frac{1}{s^2(s-3)^2} = \mathcal{L}\left(\int_0^t \frac{1}{3} x e^{3x} - \frac{1}{9} e^{3x} + \frac{1}{9}\right)$

$= \frac{1}{9} t e^{3t} - \frac{1}{27} e^{3t} + \frac{1}{27} - \left(\frac{1}{27} e^{3x}\right)\Big|_0^t + \left(\frac{1}{9} x\right)\Big|_0^t$

$= \frac{1}{9} t e^{3t} - \frac{1}{27} e^{3t} + \frac{1}{27} - \frac{1}{27} e^{3t} + \frac{1}{27} + \frac{1}{9} t - 0$

$= \frac{1}{9} t e^{3t} + \frac{2}{27} + \frac{t}{9} - \frac{2}{27} e^{3t}$

so

$y = \mathcal{L}^{-1}\left(\frac{1}{s^2(s-3)^2}\right) + \mathcal{L}^{-1}\left(\frac{1}{(s-3)^2}\right)$

$= \frac{1}{9} t e^{3t} + \frac{2}{27} + \frac{t}{9} - \frac{2}{27} e^{3t} + t e^{3t}$

$= \frac{10}{9} t e^{3t} + \frac{2}{27} e^{3t} + \frac{t}{9} + \frac{2}{27}$

4.
$y'' - 4y' + 4y = t^3 e^{2t} \quad y(0) = 0, \quad y'(0) = 0$

Laplace $\mathcal{L}(y') = s\mathcal{L}(y)$
$\mathcal{L}(y'') = s^2\mathcal{L}(y)$

$s^2\mathcal{L}(y) - 4s\mathcal{L}(y) + 4\mathcal{L}(y) = \frac{6}{(s-2)^4}$

$\cancel{(s-2)^2}\mathcal{L}(y) = \frac{6}{(s-2)^6}$

$\mathcal{L}^{-1}(y) = \mathcal{L}^{-1}\left(\frac{6}{(s-2)^6}\right)$

$y = \frac{1}{20}\mathcal{L}^{-1}\left(\frac{5!}{(s-2)^6}\right) = \frac{1}{20}e^{2t}t^5$

5. $y'' - 6y' + 13y = 0 \quad y(0) = 0 \quad y'(0) = -3$

$y_h$ $\lambda = \frac{6 \pm \sqrt{36 - 52}}{2} = \underset{\alpha}{3} \pm \underset{\beta}{2i}$

$y = c_1 e^{3t}\sin(2t) + c_2 e^{3t}\cos(2t)$

$y(0) = 0: \quad c_2 = 0$
$y'(0) = -3: \quad y' = 3c_1 e^{3t}\sin(2t)$
$+ 2c_1 e^{3t}\cos(2t)$
$+ 3c_2 e^{3t}\cos(2t)$
$- 2c_2 e^{3t}\sin(2t)$

$y'(0) = -3 = 2c_1$ since $c_2 = 0$

$c_1 = -\frac{3}{2}$ so $y = -\frac{3}{2}e^{3t}\sin(2t)$

Laplace $\mathcal{L}(y') = s\mathcal{L}(y)$
$\mathcal{L}(y'') = s^2\mathcal{L}(y) - 3$

$s^2\mathcal{L}(y) + 3 - 6\mathcal{L}(y) + 13\mathcal{L}(y) = 0$

$(s^2 - 6s + 13)\mathcal{L}(y) = -3 \qquad \frac{-3}{s^2 - 6s + 13} = \frac{-3}{2} \cdot \frac{2}{(s^2 - 6s + 9) + 4}$

$y = \mathcal{L}^{-1}\left(\frac{-3}{s^2 - 6s + 13}\right) \qquad = \frac{-3}{2}\,\frac{2}{(s-3)^2 + 2^2}$

$= -\frac{3}{2}e^{3t}\sin(2t) \qquad$ (complete the square)

6. $y'' - 5y' + 6y = u(t-1)$ $\quad y(0)=0$

$\mathcal{L}(y') = s\mathcal{L}(y)$ $\quad y'(0)=1$

$\mathcal{L}(y'') = s^2\mathcal{L}(y) - 1$

$$s^2\mathcal{L}(y) - 5s\mathcal{L}(y) + 6\mathcal{L}(y) - 1 = \frac{e^{-s}}{s}$$

$$(s^2 - 5s + 6)\mathcal{L}(y) - 1 = \frac{e^{-s}}{s}$$

$$(s-3)(s-2)\mathcal{L}(y) = \frac{e^{-s}}{s} + 1$$

$$\mathcal{L}(y) = \frac{e^{-s}}{s(s-3)(s-2)} + \frac{1}{(s-3)(s-2)} = e^{-s}\left(\frac{1}{6s} - \frac{1}{2}\frac{1}{(s-2)} + \frac{1}{3}\frac{1}{(s-3)}\right) + \frac{1}{s-3} - \frac{1}{s-2}$$

$$\frac{e^{-s}}{s(s-3)(s-2)} = \frac{A}{s} + \frac{B}{(s-2)} + \frac{C}{(s-3)} = \frac{\cancel{A(s-2)(s-3)} + B(s-3)s + C(s-2)s}{s(s-2)(s-3)}$$

$\underline{s=2}$: $B(-1)2 = 1$, $B = -\frac{1}{2}$ $\qquad$ $\underline{s=3}$: $C(1)3 = 1$, $C = \frac{1}{3}$

$\underline{s=0}$: $A(-2)(-3) = 1$, $A = \frac{1}{6}$

$$\boxed{\begin{aligned} Y = {} & u(t-1)\left[\frac{1}{6} - \frac{1}{2}e^{2(t-1)} + \frac{1}{3}e^{3(t-1)}\right] \\ & + e^{3t} + e^{2t} \end{aligned}}$$

$$\frac{1}{6s} - \frac{1}{2(s-2)} + \frac{1}{3(s-3)}$$

$$= \frac{\cancel{(s-3)(s-2)} - 3s(s-3) + 2s(s-2)}{6s(s-2)(s-3)}$$

$\cancel{s^2} - 5s + 6 - \cancel{3s^2} + 9s + \cancel{2s^2} - 4s$ ✓

✓

$$\frac{1}{(s-3)(s-2)} = \frac{A}{(s-2)} + \frac{B}{(s-3)} = \frac{A(s-3) + B(s-2)}{(s-3)(s-2)}$$

$s=3$ $\quad B = 1$

$s=2$ $\quad A = -1$

$$\frac{-1}{s-2} + \frac{1}{s-3} = \frac{-s+3+s-2}{(s-2)(s-3)} = \frac{1}{(s-2)(s-3)}$$

7.

$y' + y = \delta(t-1) \qquad y(0) = 2$

$\mathcal{L}(y') = s\mathcal{L}(y) - 2$

$s\mathcal{L}(y) - 2 + \mathcal{L}(y) = e^{-s}$

$(s+1)\mathcal{L}(y) - 2 = e^{-s}$

$\mathcal{L}(y) = \dfrac{e^{-s}}{(s+1)} + \dfrac{2}{(s+1)}$

$(\mathcal{L}(u(t-1)\, f(t-1)) = e^{-s}\mathcal{L}(f)$

$f = e^{-t} \quad \mathcal{L}(f) = \frac{1}{s+1}$

$y = u(t-1)e^{1-t} + 2e^{-t}$

8. $y'' + 16y = \delta(t-2\pi) \qquad y(0) = 0$

$y'(0) = 0$

$\mathcal{L}(y'') = s^2\mathcal{L}(y)$

$s^2\mathcal{L}(y) + 16\mathcal{L}(y) = e^{-2\pi s}$

$(s^2+16)\mathcal{L}(y) = e^{-2\pi s}$

$\mathcal{L}(y) = \dfrac{e^{-2\pi s}}{s^2+16}$

$y = u(t-2\pi)\sin(4(t-2\pi))$

$= u(t-2\pi)\sin(4t)$ by periodicity

4) $y''-4y'+4y = x^3e^{2x}$ $y(0)=1$ $y'(0)=1$

$\lambda^2-4\lambda+4=0$

$(\lambda-2)^2=0$

$\lambda=2$ double

$y_h = c_1e^{2x}+c_2xe^{2x}$

$$W = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x}+2xe^{2x} \end{vmatrix}$$

$= e^{4x}+2xe^{4x}-2xe^{4x}$

$= e^{4x}$

$$y_p = -e^{2x}\int \frac{xe^{2x}x^3e^{2x}}{e^{4x}}dx + xe^{2x}\int \frac{e^{2x}x^3e^{2x}}{e^{4x}}dx$$

$= -e^{2x}\int x^4 dx + xe^{2x}\int x^3 dx$

$= -\frac{e^{2x}x^5}{5} + \frac{x^5e^{2x}}{4}$

$\frac{5x^5e^{2x}-4x^5e^{2x}}{20} = \frac{x^5e^{2x}}{20}$

Check

$\frac{1}{5}x^5e^{2x} - \frac{4}{5}(5x^4e^{2x}+2x^5e^{2x})$

$+\frac{1}{20}(20x^3e^{2x}+10x^4e^{2x}+10x^4e^{2x}+4x^5e^{2x})$

$\frac{1}{5}x^5e^{2x} - x^4e^{2x} - \frac{2}{5}x^5e^{2x} + x^3e^{2x} + x^4e^{2x} + \frac{1}{5}x^5e^{2x}$

$= x^3e^{2x}$ ✓

$y = c_1e^{2x}+c_2xe^{2x}+\frac{x^5e^{2x}}{20}$

$y(0)=1$: $\boxed{c_1=1}$

$y'(0)=1$: $y' = 2c_1e^{2x}+c_2e^{2x}+$ ...other

$y' = 2c_1+c_2$

$\boxed{c_2 = 1-2c_1 = 1-2 = -1}$

$$\boxed{y = e^{2x}-xe^{2x}+\frac{x^5e^{2x}}{20}}$$

$\mathcal{L}(y') = s\mathcal{L}(y)-1$

$\mathcal{L}(y'') = s^2\mathcal{L}(y)-s-1$

$s^2\mathcal{L}(y)-s-1-4(s\mathcal{L}(y)-1)+4\mathcal{L}(y) = \frac{6}{(s-2)^4}$

$(s^2-4s+4)\mathcal{L}(y)-s+3 = \frac{6}{(s-2)^4}$

$(s-2)^2\mathcal{L}(y) = \frac{6}{(s-2)^4}+s-3$

$\mathcal{L}(y) = \frac{6}{(s-2)^6}+\frac{s}{(s-2)^2}-\frac{3}{(s-2)^2}$

$= \frac{1}{20}\frac{5!}{(s-2)^6}+\frac{(s-2)}{(s-2)^2}+\frac{(2-3)}{(s-2)^2}$

$= \frac{1}{20}\mathcal{L}(e^{2t}t^5)+\mathcal{L}(e^{2t})-\mathcal{L}(e^{2t}t)$

$$\boxed{= \frac{1}{20}t^5e^{2t}+e^{2t}-te^{2t}}$$