Solve the equation $x^2y' + xy = y^4$.

The general form of a Bernoulli equation is

$$y' + p(x)y = g(x)y^a$$

We can rewrite our equation as

$$y' + x^{-1}y = x^{-2}y^4,$$

so that it fits this form, with $a = 4$. The trick is to use the substitution $u = y^{1-a} = y^{-3}$.

Differentiating gives

$$u' = -3y^{-4}y',$$

and when we plug in y' and distribute, we get

$$u' = -3y^{-4}(-x^{-1}y + x^{-2}y^4) = 3x^{-1}y^{-3} - 3x^{-2}.$$

The whole point of the u-substitution was that now we have gotten rid of y in the second term, and in the first we can replace y^{-3} with u, giving

$$u' = 3x^{-1}u - 3x^{-2}$$

or

$$u' - 3x^{-1}u = -3x^{-2},$$

a first order linear differential equation, which we can solve.

If $h' = -3x^{-1}$, we can let $h = -3ln(x) = ln(x^{-3})$, and e^h becomes just x^{-3}. So we multiply through by x^{-3} to get

$$x^{-3}u' - 3x^{-4}u = -3x^{-5}$$

or

$$(x^{-3}u)' = -3x^{-5}$$

Integrating both sides gives

$$x^{-3}u = \frac{3}{4}x^{-4} + c,$$

and replacing u with y^{-3} gives

$$y = (\frac{3}{4}x^{-1} + cx^3)^{-\frac{1}{3}}.$$

This is our final answer. We can verify that we have the correct answer by finding y', and plugging y and y' back into the original equation. Notice,

$$y' = -\frac{1}{3}(\frac{3}{4}x^{-1} + cx^3)^{-\frac{4}{3}}(-\frac{3}{4}x^{-2} + 3cx^2).$$

We must verify that $x^2y' + xy = y^4$.

Plugging in, we have:

$$x^2(-\frac{1}{3}(\frac{3}{4}x^{-1} + cx^3)^{-\frac{4}{3}}(-\frac{3}{4}x^{-2} + 3cx^2)) + x((\frac{3}{4}x^{-1} + cx^3)^{-\frac{1}{3}}).$$

With a little algebra in the first term, and multiplying the second term by

$$1 = (\frac{3}{4}x^{-1} + cx^3)^{\frac{4}{3}}(\frac{3}{4}x^{-1} + cx^3)^{-\frac{4}{3}},$$

we have

$$(\frac{1}{4} - cx^4)(\frac{3}{4}x^{-1} + cx^3)^{-\frac{4}{3}} + x(\frac{3}{4}x^{-1} + cx^3)(\frac{3}{4}x^{-1} + cx^3)^{-\frac{4}{3}}$$

which simplifies to

$$(\frac{3}{4}x^{-1} + cx^3)^{-\frac{4}{3}} = y^4.$$