Ex Consider $y'' - y = \cos(x) + x$

y_h: $\lambda^2 - 1 = 0$ $\lambda = \pm 1$ $y_h = c_1 e^{x} + c_2 e^{-x}$

y_p: Guess $\underbrace{c_3 \sin(x) + c_4 \cos(x)}_{\text{from } \cos(x)} + \underbrace{c_5 x + c_6}_{\text{from } x}$

$y_p'' = -c_3 \sin(x) - c_4 \cos(x)$

$y_p'' - y = -2c_3 \sin(x) - 2c_4 \cos(x) - c_5 x - c_6 \overset{r(x)}{=} \cos(x) + x$

$c_4 = 0,\ c_5 = -1 \quad c_3 = -\frac{1}{2} \quad c_6 = 0$

$y_p = -\frac{1}{2}\cos(x) - 1 = -(\frac{1}{2}\cos(x) + x)$

$y = \underbrace{c_1 e^{x} + c_2 e^{-x}}_{y_h} \underbrace{- (\frac{1}{2}\cos(x) + x)}_{y_p}$

Ex (Breaking Rules)

$y'' + y = \cos(x)$

y_h: $\lambda^2 + 1 = 0$ $\lambda = \pm i$ $y_h = c_1 \cos(x) + c_2 \sin(x)$

notice $\cos(x)$ appears in y_h; so should choose $y_p = c_3 x \cos(x) + c_4 x \sin(x)$.

suppose we don't multiply through by x?

Try: $y_p = c_3 \cos(x) + c_4 \sin(x)$

$y_p'' = -c_3 \cos(x) - c_4 \sin(x)$ X $y_p'' + y_p = 0$! (Because these are already solutions to the homogeneous equation.)

so choose $y_p = c_3 x \cos(x) + c_4 x \sin(x)$

then $y_p' = c_3 \cos(x) - c_3 x \sin(x) + c_4 \sin(x) + c_4 x \cos(x)$

$y_p'' = -c_3 \sin(x) - c_3 \sin(x) - c_3 x \cos(x) + c_4 \cos(x) + c_4 \cos(x) - c_4 x \sin(x)$

$= -2c_3 \sin(x) - c_3 x \cos(x) + 2c_4 \cos(x) - c_4 x \sin(x)$

$y_p'' + y_p = -2c_3 \sin(x) + 2c_4 \cos(x) \overset{r(x)}{=} \cos(x)$ $\boxed{c_4 = \frac{1}{2} \quad c_3 = 0}$

$\hookrightarrow y_p = \frac{1}{2} x \sin(x)$

Verify

$(\frac{1}{2} x \sin(x))'' = (\frac{1}{2}\sin(x) + \frac{1}{2} x \cos(x))'$

$= (\frac{1}{2}\cos(x) + \frac{1}{2}\cos(x) - \frac{1}{2} x \sin(x))$ ✓