A general method for solving ϵ-δ proofs.

To prove that

$$\lim_{x \to a} f(x) = L$$

0. It's good to start by writing out everything you know about the problem. List a, L, $f(x)$, $|f(x) - L|$, $|x - a|$, and the domain of f.

1. Make sure a is an accumulation point of the domain of f. State this fact and give a good explanation. If the domain is $\mathbb{R}$, then stating that is enough. Otherwise, if a is an interior point, or a right or left endpoint of an interval in the domain; state that.

2. State "Let $\epsilon > 0$ be given." You are going to find a function of ϵ which gives you a δ that satisfies the definition.

3. Try to factor $|x - a|$ out of $|f(x) - L|$. The result should be something that looks like $(g(x))(|x - a|)$. If $g(x)$ is just some constant number, you're practically done: see bottom of page. If it is a function, there is more to do. (Note, the function should always be positive). This part often requires things like combining fractions, or multiplying by the conjugate.

4. Choose a δ_1. This is pretty much arbitrary, meaning it often doesn't matter what number you pick (1 is usually a good choice). If your domain is not $\mathbb{R}$, though, try to pick a δ_1 so that no number in

$$(a - \delta_1, a) \cup (a, a + \delta_1)$$

is nearby any potentially annoying parts of the domain. For example, if $a = 1$, and the domain of f is $(0, \infty)$, it's a good idea to let $\delta_1 = \frac{1}{2}$ rather than 1.

5. Now find some M so that $g(x) < M$ for every x in $(a - \delta_1, a) \cup (a, a + \delta_1)$. I.e. whenever $0 < |x - a| < \delta_1$, $g(x) < M$. Note that you can do steps 4 and 5 backwards if your careful; you can choose an M which you know will work for some δ_1, but make sure you provide valid mathematical reasoning. Also, remember you don't necessarily need the *smallest* M; any value which you can prove is greater than all values of $g(x)$ when $0 < |x - a| < \delta_1$ will work.

6. Show a chain of inequalities as follows:

$$|f(x) - L| = g(x)|x - a| < M|x - a| \text{ whenever } 0 < |x - a| < \delta_1$$

   This is because $g(x) < M$ when $0 < |x - a| < \delta_1$. And for any δ_2,

$$|f(x) - L| = g(x)|x - a| < M|x - a| < M\delta_2$$

   whenever $0 < |x - a| < \delta_1$ and $0 < |x - a| < \delta_2$.

7. So let $\delta_2 = \frac{\epsilon}{M}$; and let $\delta = \min\{\delta_1, \delta_2\}$. Remember, δ_1 is the arbitrary value that you've chosen, and δ_2 is a function of ϵ which you were able to find after you had chosen δ_1.

8. Now we have that $|f(x) - L| < \epsilon$ whenever $|x - a| < \delta$, which is what we wanted to prove.

With each step, you should be writing something out in the proof. In part 3, you should be showing the algebra you use to factor; in part 4 you should say something like "Let $\delta_1 = 1$" or "Consider x such that $|x-a| < 1$"; etc. Remember, when we do this, we are restricting the values of x that we are looking at, and the chain of inequalities in part 6 is only true with these restrictions, so make sure you state these facts when they are relevant (i.e. in part 6).

If in step 3 you have some $M|x - a|$ for some constant; then you can let $\delta = \frac{\epsilon}{M}$, and show that the inequality works:

$$|f(x) - L| = M|x - a| < M\delta \text{ whenever } |x - a| < M$$

But since $\delta = \frac{\epsilon}{M}$, $M\delta = \epsilon$; and we have that $|f(x) - L| < \epsilon$, which is what we wanted; so we're done. You should still sum it up with step 8 though.

If you have questions, feel free to e-mail me: kjones@math.binghamton.edu