The Bieri-Neumann-Strebel Invariant and Controlled Connectivity

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The Bieri-Neumann-Strebel (BNS) Invariant Σ^1

For a finitely generated group G, $\Sigma^1(G)$ is an open subspace of a sphere which records information about the "connectivity of G with respect to direction."

- Study "connectivity of G" via a geometric proxy – the Cayley graph.

- Need to impose a sense of direction on G.

Directions in G I - A natural vector space

Let $V_G = G_{ab} \otimes \mathbb{R}$

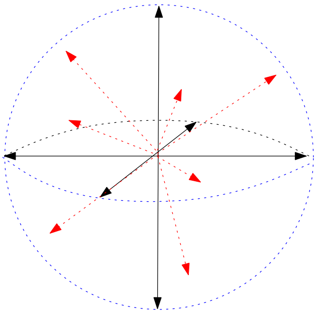
- a finite dimensional real vector space on which G naturally acts by translations.

- provides a mechanism for imposing a sense of direction on G, as follows:

Directions in G II - The sphere at infinity

Let $S_\infty = \{\tau \subset V_G \mid \tau \text{ is a ray emanating from the origin}\}$

(i.e., τ is an isometric embedding $[0, \infty) \to V_G$ with $\tau(0) = \vec{0}$)

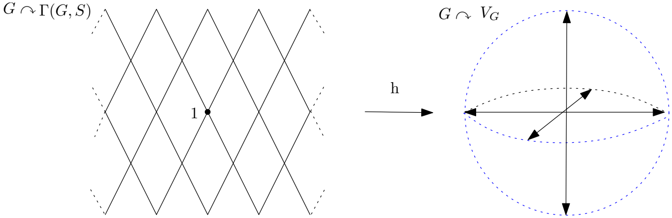


Directions toward which one can "look to infinity" in V_G

Directions in G III - Lifting directions

Choose a finite generating set S for G and let $\Gamma(G,S)$ be the Cayley graph of G with respect to S.

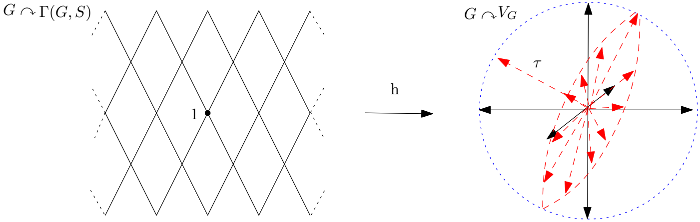
There exists a continuous G-equivariant *control map* $h : \Gamma(G,S) \to V_G$



Directions in G III - Lifting directions

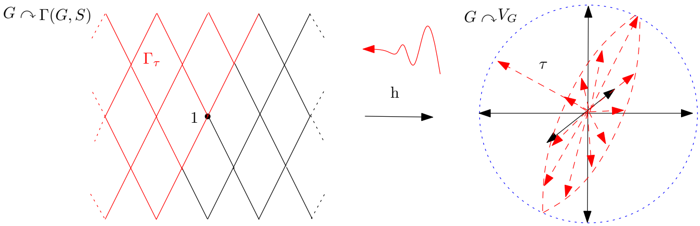
Place an inner product $\langle \cdot, \cdot \rangle$ on V_G.

For a ray $\tau \in S_\infty$, set $H_\tau = \{\vec{v} \in V_G \mid \langle \tau, \vec{v} \rangle \geq 0\}$, the "half-space" in the direction of τ.



Directions in G III - Lifting directions

Set Γ_τ be the largest subgraph of $h^{-1}(H_\tau)$, the "half-graph" in the direction of τ.



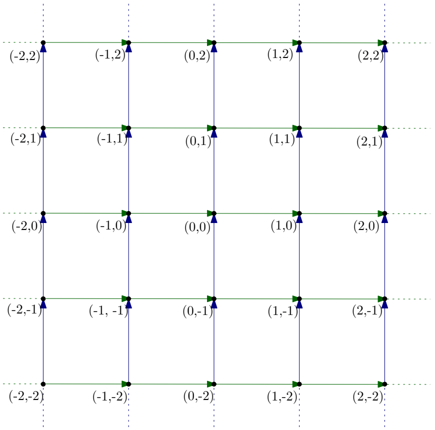
Definition of $\Sigma^1(G)$

If Γ_τ is connected, we say G is *connected in the direction of* τ.

Define the Bieri-Neumann-Strebel (BNS) invariant

$$\Sigma^1(G) = \{\tau \mid G \text{ is connected in the direction of } \tau\}$$

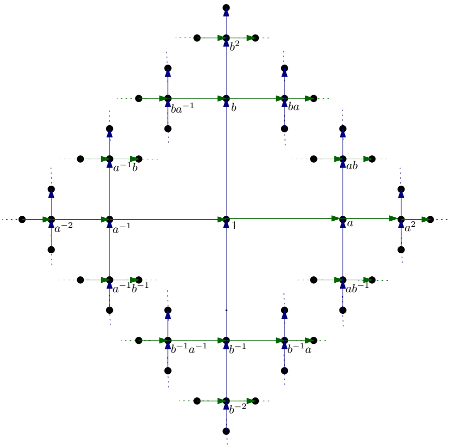
Example I - $\mathbb{Z}^2$ (one extreme)



$\Sigma^1(\mathbb{Z}^2) = S_\infty(\mathbb{Z}^2) \cong S^1$

(For any finitely generated abelian group, $\Sigma^1 = S_\infty$)

Example II - The free group F(a,b) (the other extreme)



$\Sigma^1(F(a,b)) = \emptyset$

- Cayley graph Γ is a tree

- For any free group, $\Sigma^1 = \emptyset$

Applications

- (Bieri-Neumann-Strebel, 1987) $\Sigma^1(G)$ characterizes the set of finitely generated normal subgroups of G with abelian quotient:
  - Given $N \triangleleft G$ with G/N abelian, there is a minimal N invariant subspace $V_N \subseteq V_G$.
  - N is finitely generated if the directions *orthogonal* to V_N are in $\Sigma^1(G)$.
- (Brown, 1987) $\Sigma^1(G)$ characterizes HNN decompositions of G over finitely generated base groups.
- (Bieri-Renz, 1988) Analagous invariants characterize higher dimensional topological and homological finiteness properties.

Controlled Connectivity

Bieri and Geoghegan - the action $G \curvearrowright G_{ab} \otimes \mathbb{R}$ can be replaced with any action $G \overset{\rho}{\curvearrowright} M$ that is "nice enough".

$\Sigma^1(G)$ is a special case of the Bieri-Geoghegan invariant $\Sigma^1(\rho)$.

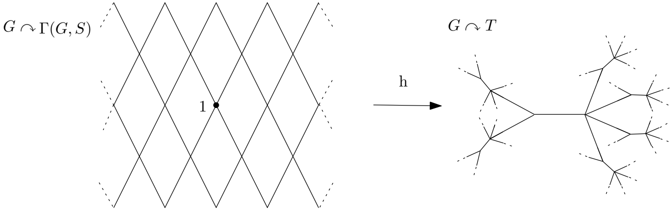
Nice Enough:

1. M is a CAT(0) metric space.
   - CAT(0) is a purely metric notion of nonpositive curvature.
   - A CAT(0) space are contractible.
   - A CAT(0) space has a boundary at infinity $\partial_\infty M$, analagous to the sphere at infinity.
2. M is proper; that is, the closure of any metric ball is compact.
   - This guarantees that $\partial_\infty M$ is a compact topological space (though not necessarily a sphere).
3. ρ is by isometries.

Definition of $\Sigma^1(\rho)$

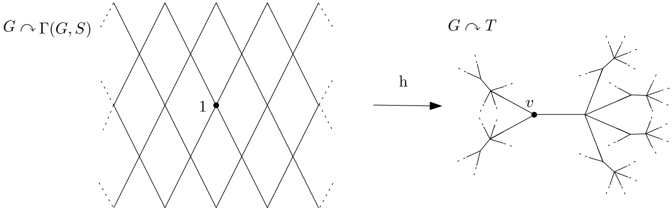
(Case of $M = T$, a locally finite tree.)

As before, we can choose a control map $h : \Gamma(G, S) \to T$.



Definition of $\Sigma^1(\rho)$

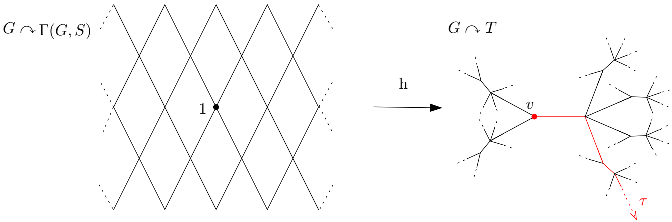
Fix a base vertex v.



Definition of $\Sigma^1(\rho)$

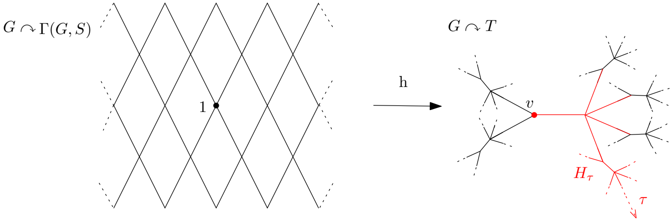
Then

$$\partial_\infty T \cong \{\tau \subseteq T \mid \tau \text{ is a ray emanating from } v\}$$



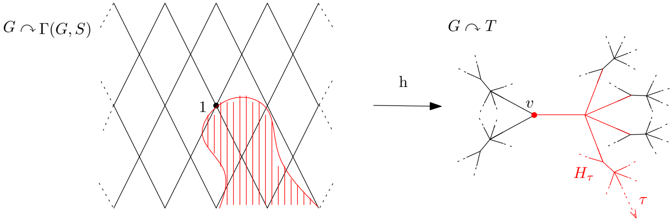
Defintion of $\Sigma^1(\rho)$

There is a standard notion of a neighborhood (in T) about τ, called a horoball H_τ.



Definition of $\Sigma^1(\rho)$

Define Γ_τ to be the preimage $h^{-1}(H_\tau)$.



Definition of $\Sigma^1(\rho)$

Say ρ is controlled connected over τ if $\Gamma_\tau := h^{-1}(H_\tau)$ is connected. (This depends only on ρ.)

Define

$$\Sigma^1(\rho) = \{\tau \mid \rho \text{ is controlled connected over } \tau\}$$

Calculations

- (Bieri-Geoghegan, 2004) Modular group a acting on hyperbolic plane
- (Rehn, 2008) Arithmetic groups acting on symmetric spaces
- (J, 2009) Actions on trees which factor through quotient groups

The Sphere at Infinity, S_∞

The Little Prince can see all directions from his tiny planet.

Thank you!