A Geometric Invariant of Discrete groups

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MAA Seaway Section, Fall ’09

Outline

Groups and Cayley Graphs

A Very Nice Theorem

Directions in the Cayley graph

The BNS-Invariant $\Sigma^1(G)$

Finitely Generated Groups

Definition

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The group G is *finitely generated* if there exists a finite generating set for G.

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For a group G and a fixed generating set S, let $\Gamma = \Gamma(G, S)$ be the graph with vertices and edges:

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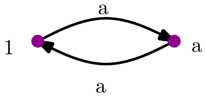
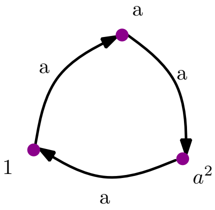
- Has "loops" only if $1 \in S$.
- Circuits correspond to trivial words (i.e., relations).

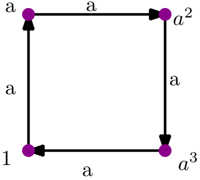
Cayley graphs of cyclic groups



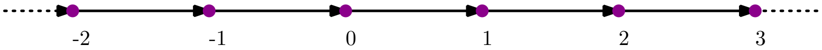
C_2 (Cyclic, order 2)



C_3, (Cyclic, order 3)

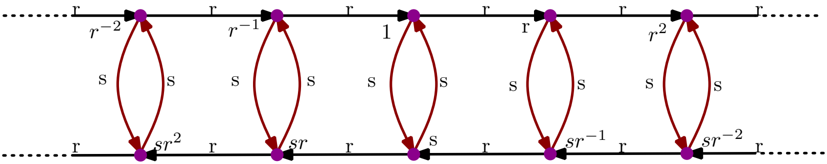


C_4 (Cyclic, Order 4)



$\mathbb{Z} \cong C_\infty$ (Cyclic, order ∞)

Two Cayley graphs for the infinite dihedral group D_∞



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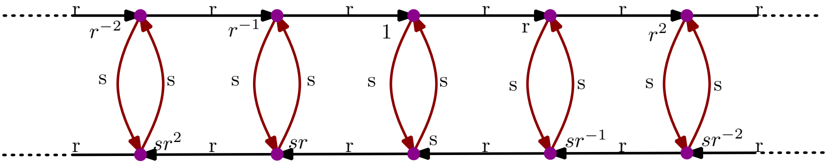

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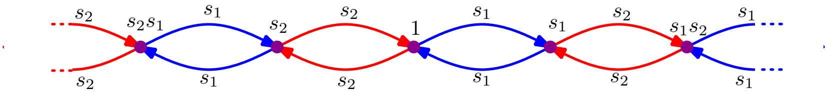

$G = D_\infty$ $S = \{s_1, s_2\}$ such that $|s_1| = |s_2| = 2$ and s_1 and s_2 satisfy no relations.

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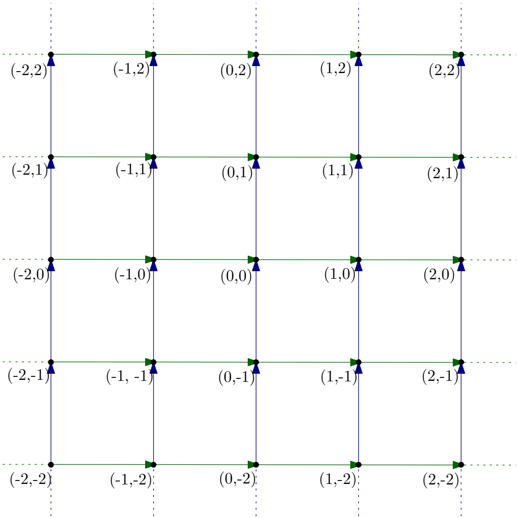
Motto: "Coarse Geometry"

Other interesting Cayley graphs: $\mathbb{Z}^2$ (free abelian group)



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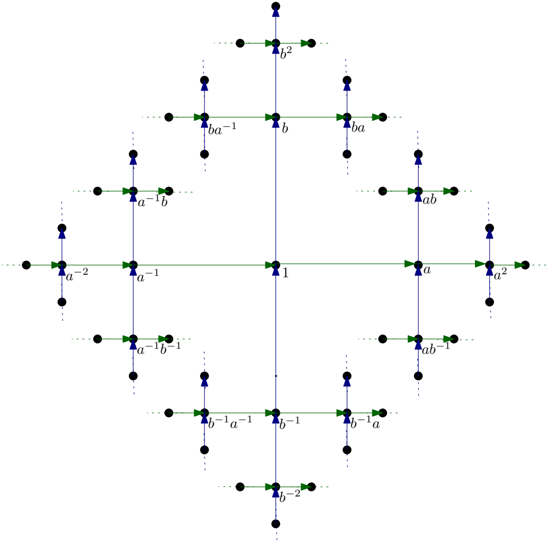

The Cayley graph of a free group is a tree

Outline

A Very Nice Theorem

Theorem (A)

Let G be a group acting on a connected graph Γ such that the stabilizer of any point is finitely generated. If the quotient graph $G \backslash \Gamma$ is finite, then G is finitely generated.

(Appears in Serre's *Trees*.) Is it older?

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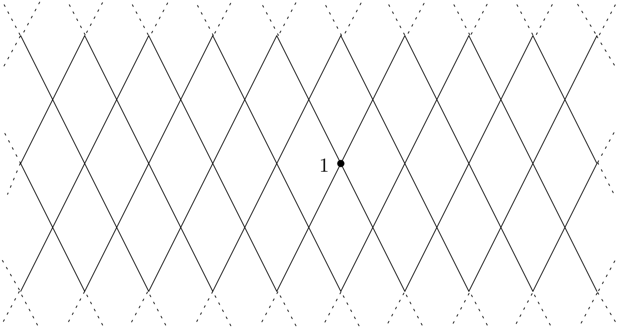
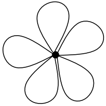
So G acts on its Cayley graph with trivial point stabilizers and finite quotient graph. (So what? We assumed G was finitely generated to begin with!)

Γ



$G\backslash\Gamma$



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(But so what!? This can be proved with pure group theory!)

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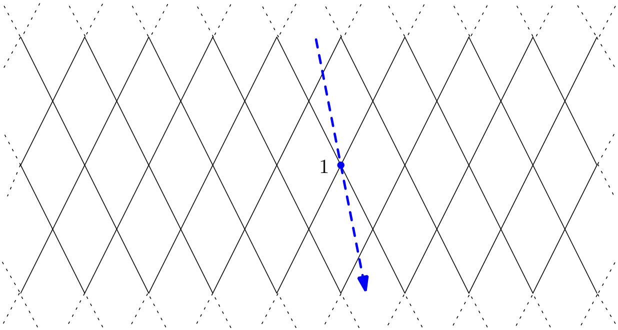
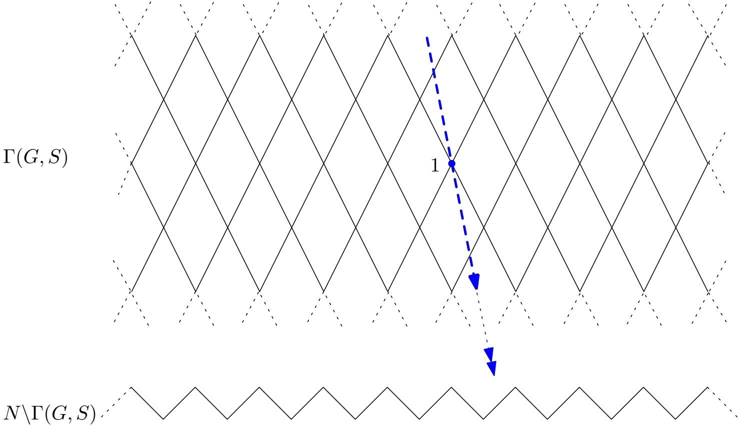
Outline

We want to impart a sense of *direction* on our Cayley graph.

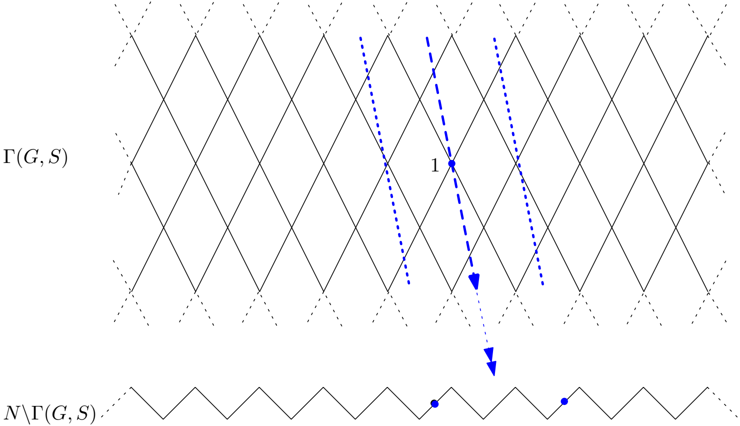
If N translates Γ in a certain set of fixed directions...



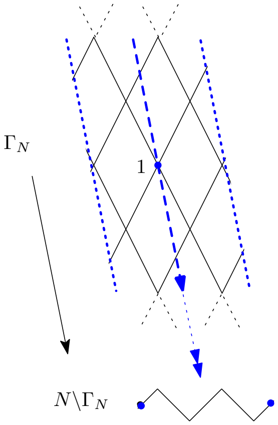
...then even though the quotient is not finite ...



...we can take a "strip" of Γ...



...on which N acts with finite quotient.



Two Problems

Problem 1: How do we impart a sense of direction on the Cayley graph?

Problem 2: How do we know if a strip is connected?

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A vector space has a sense of direction:

Take rays eminating from the origin.

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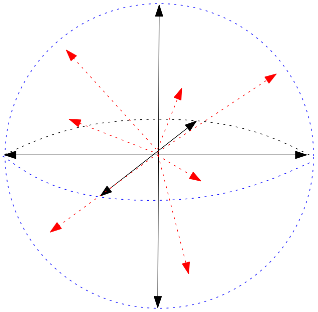

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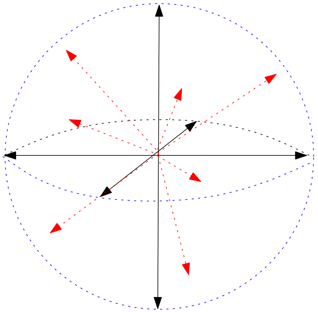

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The set of directions forms a sphere.

The Sphere at Infinity, S_∞

The Little Prince can see all directions from his tiny planet.

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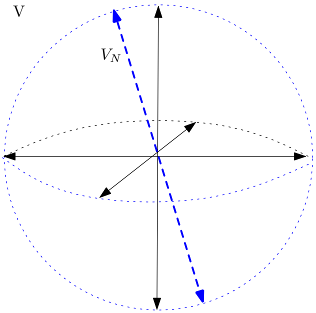

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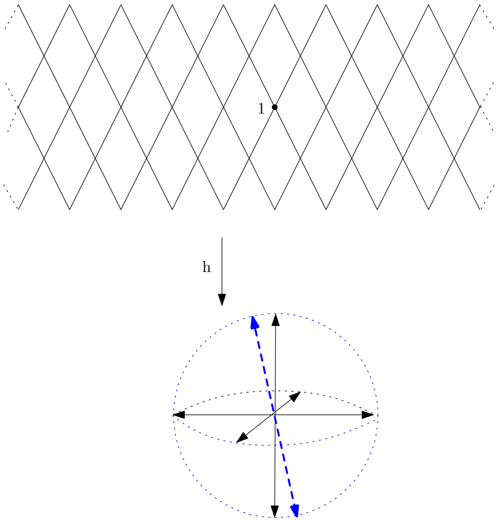

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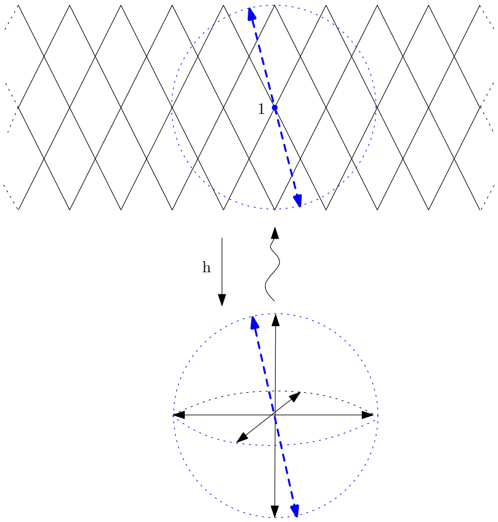

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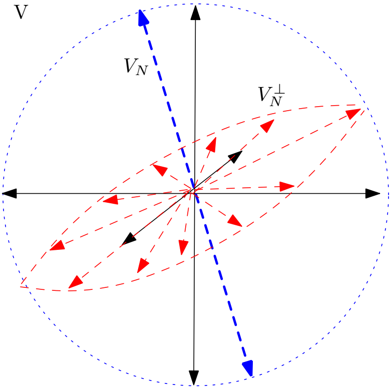
$$\Sigma^1(G) = \{\vec{u} \mid G \text{ is connected in the direction of } \vec{u}\}$$

Let $S_N^\perp$ be the set of directions orthogonal to V_N.



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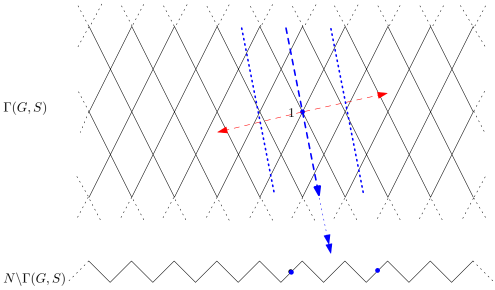

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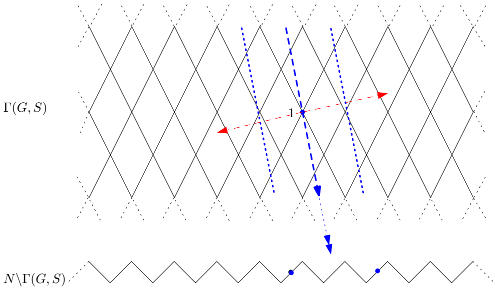

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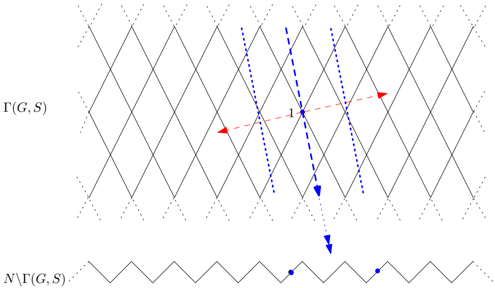

Then we can find a strip wide enough that is connected.
This is technical.

- Uses the fact that a sphere is compact.
- Uses the fact that G is finitely generated
- Uses the fact that $\Sigma^1(G) \subset S_\infty$ is *open*.

Theorem (Bieri-Neumann-Strebel)

Let $N \triangleleft G$ such that G/N is abelian. Then N is finitely generated if and only if $S_N^{\perp} \subseteq \Sigma^1(G)$.

In particular the commutator subgroup is finitely generated if and only if $\Sigma^1(G) = S_\infty$.