

[20] 1. Solve for x :

(i) $\ln(5 \cdot 2^x) = e^{(1+\ln x)}$

$$\ln 5 + \ln 2^x = e \cdot e^{\ln x}$$

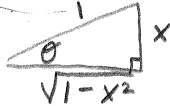
$$\ln 5 + x \ln 2 = e^x$$

$$\ln 5 = (e - \ln 2)x$$

$$x = \frac{\ln 5}{e - \ln 2}$$

(ii) $\tan^2(\sin^{-1} x) = 4$

$$\sin^{-1} x = \theta$$



$$\tan^2(\theta) = 4$$

$$\left(\frac{x}{\sqrt{1-x^2}}\right)^2 = 4$$

$$\frac{x^2}{1-x^2} = 4$$

$$x^2 = 4(1-x^2)$$

$$5x^2 = 4$$

$$x^2 = \frac{4}{5}$$

$$x = \pm \frac{2}{\sqrt{5}}$$

[40] 2. Calculate the derivative for each of the following. You need not simplify.

(i) $f(x) = \cosh\left(\frac{1}{x+e^x}\right)$

$$f'(x) = \sinh\left(\frac{1}{x+e^x}\right) \cdot (-1)(x+e^x)^{-2} \cdot (1+e^x)$$

(ii) $g(x) = (\tan^{-1} x)^{\sqrt{2}}$

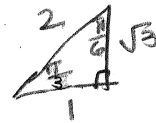
$$g'(x) = \sqrt{2} (\tan^{-1} x)^{\sqrt{2}-1} \left(\frac{1}{1+x^2}\right)$$

(iii) $h(x) = (x^2 + 1)^{\sin^{-1} x}$

$$h'(x) = (x^2+1) \left(\frac{1}{\sqrt{1-x^2}}\right) + 2x \sin^{-1} x$$

(iv) $j(x) = \sum_{n=1}^{\infty} \frac{(\cos n)x^{2n+1}}{n!}$

$$j'(x) = \sum_{n=1}^{\infty} \frac{(\cos n)(2n+1)x^{2n}}{n!}$$



$$u = \tan x \\ du = \sec^2 x dx$$

[80] 3. Evaluate each of the following integrals by any method.

$$\begin{aligned} \text{(i)} \int_0^{\pi/3} \tan^4 x dx &= \int_0^{\pi/3} (\sec^2 x - 1) \tan^2 x dx = \int_0^{\pi/3} \sec^2 x \tan^2 x dx - \int_0^{\pi/3} \tan^2 x dx \\ &= \int_0^{\sqrt{3}} u^2 du - \int_0^{\pi/3} (\sec^2 x - 1) dx = \left[\frac{u^3}{3} \right]_0^{\sqrt{3}} - [\tan x - x]_0^{\pi/3} \\ &= \frac{3\sqrt{3}}{3} - \left(\sqrt{3} - \frac{\pi}{3} \right) = \frac{\pi}{3} \end{aligned}$$

$$\begin{aligned} \text{(ii)} \int \frac{x^4}{x^3 + 2x} dx & \quad \begin{array}{l} x^3 + 2x \sqrt{x^4} \\ -x^4 - 2x^2 \\ \hline -2x^2 \end{array} \\ &= \int \left(x + \frac{-2x^2}{x^3 + 2x} \right) dx \\ &= \frac{1}{2} x^2 - \int \frac{2x^2}{x(x^2 + 2)} dx = \frac{1}{2} x^2 - \int \frac{x}{x^2 + 2} dx \\ & \quad \begin{array}{l} u = x^2 + 2 \\ du = 2x dx \end{array} \\ \text{(iii)} \int \ln x dx &= \frac{1}{2} x^2 - \int \frac{1}{2u} du \\ &= \frac{1}{2} x^2 + \frac{1}{2} \ln |x^2 + 2| + C \end{aligned}$$

$$\begin{aligned} u &= \ln x \quad dv = dx \\ du &= \frac{1}{x} dx \quad v = x \\ x \ln x - \int dx &= x \ln x - x + C \end{aligned}$$

$$\begin{aligned} \text{(iv)} \int_0^1 x 2^{x^2} dx & \quad \begin{array}{l} u = x^2 \quad du = 2x dx \\ \frac{1}{2} du = x dx \end{array} \\ &= \frac{1}{2} \int_0^1 2^u du = \frac{1}{2} \left[\frac{2^u}{\ln 2} \right]_0^1 = \frac{1}{2} \left(\frac{2}{\ln 2} - \frac{1}{\ln 2} \right) = \frac{1}{2 \ln 2} = \frac{1}{\ln 4} \\ & \quad \text{or } \log_4 e \end{aligned}$$

(v) $\int e^{x^2} dx$ Not an elementary function, so use power series.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \text{so } e^{x^2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$$

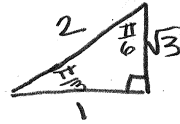
$$\int \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} dx = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{n!(2n+1)} + C$$

[80] 4. Evaluate each of the following limits.

(i) $\lim_{x \rightarrow 0} \frac{\sin(3 \sin 2x)}{x} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{\cos(3 \sin 2x) \cdot 6 \cos 2x}{1} = \frac{1 \cdot 6}{1} = 6$

(ii) $\sum_{n=1}^{\infty} \frac{(-1)^n (\frac{\pi}{6})^{2n}}{(2n)!} = \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$

Maclaurin series of $\cos x$ at $\frac{\pi}{6}$



(iii) $\lim_{x \rightarrow 0} (1 - 3x)^{2/x} = \lim_{x \rightarrow 0} e^{\frac{2}{x} \ln(1 - 3x)}$

$\lim_{x \rightarrow 0} \frac{2 \ln(1 - 3x)}{x} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{-6}{1 - 3x} = -6$

$\therefore \lim_{x \rightarrow 0} (1 - 3x)^{2/x} = e^{-6}$

(iv) $\sum_{n=0}^{\infty} [\arctan(n+2) - \arctan(n)]$ Telescoping series

$S_n = \sum_{k=0}^n (\arctan(k+2) - \arctan(k)) = (\arctan 2 - \arctan 0) + (\arctan 3 - \arctan 1) + (\arctan 4 - \arctan 2) + \dots + (\arctan n - \arctan(n-2)) + (\arctan(n+1) - \arctan(n-1)) + (\arctan(n+2) - \arctan(n))$

So $\lim_{n \rightarrow \infty} S_n = -\arctan 1 + \lim_{n \rightarrow \infty} \arctan(n+1) + \lim_{n \rightarrow \infty} \arctan(n+2)$

(v) $\int_0^{\infty} \frac{1}{2x^2 + 3x + 1} dx = \star$

$\frac{1}{(2x+1)(x+1)} = \frac{A}{2x+1} + \frac{B}{x+1}$

$1 = A(x+1) + B(2x+1)$
 $1 = (A+B)x + A+B$
 $A+B=0$ and $A+B=1$
 $A-2=0$ $-2B+B=1$
 $A=2$ $B=-1$

$\int_0^{\infty} (\frac{2}{2x+1} - \frac{1}{x+1}) dx$
 $= \lim_{t \rightarrow \infty} \int_0^t (\frac{2}{2x+1} - \frac{1}{x+1}) dx$
 $= \lim_{t \rightarrow \infty} [\ln|2x+1| - \ln|x+1|]_0^t$
 $= \lim_{t \rightarrow \infty} [\ln|2t+1| - \ln|t+1|]$
 $= \lim_{t \rightarrow \infty} \ln \left| \frac{2t+1}{t+1} \right| = \ln \left(\lim_{t \rightarrow \infty} \frac{2t+1}{t+1} \right) = \ln 2$

[30] 5. Find the radius and interval of convergence for the series $\sum_{k=1}^{\infty} \frac{2^k (x-1)^{2k}}{\sqrt{k}}$.

Using
Ratio
Test

$$\lim_{k \rightarrow \infty} \left| \frac{2^{k+1} (x-1)^{2k+2}}{\sqrt{k+1}} \cdot \frac{\sqrt{k}}{2^k (x-1)^{2k}} \right|$$

$$= \lim_{k \rightarrow \infty} \left| \frac{2(x-1)^2 \sqrt{k}}{\sqrt{k+1}} \right| = 2|x-1|^2 < 1 \text{ when the sum converges}$$

$$\text{So } |x-1|^2 < \frac{1}{2}$$

$$|x-1| < \frac{1}{\sqrt{2}}$$

$$-\frac{1}{\sqrt{2}} < x-1 < \frac{1}{\sqrt{2}}$$

$$-\frac{1}{\sqrt{2}} + 1 < x < \frac{1}{\sqrt{2}} + 1$$

Thus $R = \frac{1}{\sqrt{2}}$ and
 $I = \left(-\frac{1}{\sqrt{2}} + 1, \frac{1}{\sqrt{2}} + 1 \right)$

Check endpts

① $\sum_{k=1}^{\infty} \frac{2^k \left(\frac{1}{\sqrt{2}}\right)^{2k}}{\sqrt{k}} = \sum_{k=1}^{\infty} \frac{2^k \left(\frac{1}{2^k}\right)}{\sqrt{k}} = \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ diverges (p-series with $p = \frac{1}{2} \leq 1$)

② $\sum_{k=1}^{\infty} \frac{2^k \left(\frac{1}{\sqrt{2}}\right)^{2k}}{\sqrt{k}} = \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ same diverges

[40] 6. Find Maclaurin series for each of the following.

(i) $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

(ii) $\frac{x^3}{1+x^4} = x^3 \sum_{n=0}^{\infty} (-1)^n x^{4n} = \sum_{n=0}^{\infty} (-1)^n x^{4n+3}$

(iii) $\ln|1+x^4| = 4 \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{4n+4} = 4 \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{4n+4}$

$\frac{d}{dx} \ln|1+x^4| = \frac{4x^3}{1+x^4} = 4 \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{4n+4}$

Plugging in $x=0$, we see $c=0$.

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 4x^{4n+3}}{4n+4}$$

(iv) $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

(v) $\frac{d(x^2 \sin x)}{dx} = \frac{d}{dx} \left(x^2 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \right) = \frac{d}{dx} \left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+3}}{(2n+1)!} \right)$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (2n+3) x^{2n+2}}{(2n+1)!}$$

[30] 7. Let $f(x) = \sinh(x)$

(i) Calculate $T_5(x)$, the fifth degree Taylor polynomial for $f(x)$ (expanded at $a = 0$).

n	$f^{(n)}(x)$	$f^{(n)}(a)$
0	$\sinh(x)$	0
1	$\cosh x$	1
2	Repeats	0
3		1
4		0
5		1

$$\sum_{n=0}^5 \frac{f^{(n)}(0)}{n!} x^n = x + \frac{x^3}{3!} + \frac{x^5}{5!}$$

(ii) Use the Taylor's Inequality to bound the error in estimating $\sinh(1)$ with $T_5(1)$.

Taylor's Inequality tells us if $|f^{(n+1)}(x)| \leq M$ for $|x-a| \leq d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$|R_n(x)| \leq \frac{M}{(n+1)!} |x-a|^{n+1} \quad \text{for } |x-a| \leq d$$

We care about $n=5$. Well $f^{(6)}(x) = \sinh(x)$.
 $y = \sinh x$ is an increasing function, so takes its highest value on $[-1, 1]$ at 1. Furthermore, since $y = \sinh x$ is odd, $y = 1 \sinh x$ takes its highest value at ± 1 , and at both these points equals $\sinh 1$. Thus

$$|\sinh x| \leq \sinh 1 \quad \text{for } x \in [-1, 1]$$

Therefore, by Taylor's inequality,

$$|R_5(x)| \leq \frac{\sinh 1}{6!} |x|^6 \leq \frac{e^{-1/e}}{2(6!)} |1|^6 = \frac{e^{-1/e}}{2(6!)} \quad \text{for } |x| \leq 1$$

Thus the error of approximating $\sinh(1)$ with $T_5(1)$ is at most

$$\frac{e^{-1/e}}{2(6!)}.$$

[40] 8. Let $g(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{n!}$.

(i) Calculate the domain of g (i.e. the interval of convergence).

By ratio test converges if $\lim_{n \rightarrow \infty} \left| \frac{x^{2n+3}}{(n+1)!} \cdot \frac{n!}{x^{2n+1}} \right|$
 $= \lim_{n \rightarrow \infty} \left| \frac{x^2}{n+1} \right| = 0 < 1$ which is always true.

Hence $I = (-\infty, \infty)$

(ii) Show that g is a strictly increasing function and consequently has an inverse function.

$$g'(x) = \sum_{n=0}^{\infty} \frac{(2n+1)x^{2n}}{n!} = \sum_{n=0}^{\infty} \frac{(2n+1)(x^2)^n}{n!} \text{ always } > 0 \text{ for } x \neq 0$$

$\Rightarrow g'(x) > 0$ for $x \neq 0$. When $x=0$, $g'(0)=1$ is also > 0 . So g is strictly increasing.

(iii) Calculate $g(1)$.

$$g(1) = \sum_{n=0}^{\infty} \frac{1^{2n+1}}{n!} = \sum_{n=0}^{\infty} \frac{1^n}{n!} = e$$

(iv) Calculate $[(g^{-1})'(e)]$.

$$(g^{-1})'(e) = \frac{1}{g'(g^{-1}(e))} = \frac{1}{g'(1)} = \frac{1}{\sum_{n=0}^{\infty} \frac{(2n+1)}{n!}} = \frac{1}{\sum_{n=0}^{\infty} \frac{2n}{n!} + \sum_{n=0}^{\infty} \frac{1}{n!}}$$

$$= \frac{1}{2 \sum_{n=1}^{\infty} \frac{1}{n!} + e} = \frac{1}{2e + e} = \frac{1}{3e}$$

$\sum_{n=0}^{\infty} \frac{2n}{n!} = 0 + 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots$
 $= \sum_{n=0}^{\infty} \frac{1}{n!} = e$

[12] 9. (i) Find polar coordinates for the point whose cartesian coordinates are $(1, -1)$.

$$\tan \theta = -1 \quad \theta = -\frac{\pi}{4}$$

$$r^2 = x^2 + y^2 \quad r = \pm \sqrt{2}$$

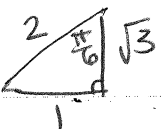
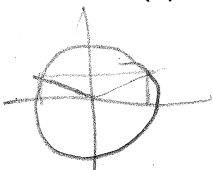
$$\left(-\frac{\pi}{4}, \sqrt{2} \right)$$

(ii) Find the Cartesian coordinates for the point with polar coordinates $(-1, \frac{5\pi}{6})$.

$$x = -1 \cos \frac{5\pi}{6} = \frac{\sqrt{3}}{2}$$

$$\left(\frac{\sqrt{3}}{2}, \frac{1}{2} \right)$$

$$y = -1 \sin \frac{5\pi}{6} = -\frac{1}{2}$$



[30] 10. Let C be the graph of those points whose polar coordinates satisfy the equation

$$r = \theta, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

(i) Calculate the slope of the line tangent to C at the point where $\theta = \frac{\pi}{4}$.

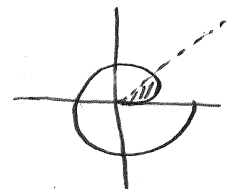
$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta} \sin\theta + r \cos\theta}{\frac{dr}{d\theta} \cos\theta - r \sin\theta} = \frac{\sin\theta + \theta \cos\theta}{\cos\theta - \theta \sin\theta}$$

$$\left. \frac{dy}{dx} \right|_{\frac{\pi}{4}} = \frac{\sin\frac{\pi}{4} + \frac{\pi}{4} \cos\frac{\pi}{4}}{\cos\frac{\pi}{4} - \frac{\pi}{4} \sin\frac{\pi}{4}} = \frac{\frac{1}{\sqrt{2}} + \frac{\pi}{4} \frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}} - \frac{\pi}{4} \frac{1}{\sqrt{2}}} = \boxed{\frac{1 + \frac{\pi}{4}}{1 - \frac{\pi}{4}}}$$

(ii) Set up the integral for the area of the region in the first quadrant bounded by $r = \theta$

and $\theta = \frac{\pi}{4}$. [No credit for integrating it]

$$\int_0^{\pi/4} \frac{1}{2} \theta^2 d\theta$$



(iii) Set up the integral for the arc length of C in the first quadrant. [no credit for integrating

it]

$$\int_0^{\pi/2} \sqrt{\theta^2 + 1} d\theta$$

[18] 11. On the axes to the right identify and sketch the conic section whose Cartesian

coordinates satisfy $x^2 + 2x - y^2 - 4y = 4$.

$$x^2 + 2x + 1 - 1 - y^2 - 4y + 4 - 4 = 4$$

$$(x+1)^2 - (y-2)^2 = 9$$

$$\frac{(x+1)^2}{9} - \frac{(y-2)^2}{9} = 1 \quad a=b=3$$

$$c = \sqrt{18} = 3\sqrt{2}$$

Type of conic hyperbola

opens left/right

foci $(\pm 3\sqrt{2}, 0)$

vertices $(\pm 3, 0)$

Asymp $y = \pm x$

