QR = Question removed, so please ignore

Test 2

1. (a) $\lim_{n\to\infty} S_n = S$

(b) $\{a_n\}$ is bounded above and below.

(c) $a_{n+1} < a_n$ for all $n \in \mathbb{N}$.

2. (a) $a_1 = \frac{3}{3!} = \frac{3}{6} = \frac{1}{2}$ $\quad a_2 = \frac{3^2}{4!} = \frac{9}{24}$

decreasing

Want to show

$a_{n+1} < a_n \quad \forall n \in \mathbb{N}$

$$\frac{3^{n+1}}{(n+3)!} < \frac{3^n}{(n+2)!}$$

$$\frac{3\,(n+2)!}{(n+3)!} < 1$$

$$\frac{3}{(n+3)} < 1$$

$3 < n+3$

$n > 0$. Always true.

Thus we can start from here and work backwards.

QR (b) $\lim_{n\to\infty} \frac{3^n}{(n+2)!}$

$$a_n = \frac{3\cdot 3\cdot 3\cdot 3\cdot 3\cdot 3 \ldots 3\cdot 3\cdot 3\cdot 3}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6 \ldots (n-3)(n-2)(n-1)\,n\,(n+1)(n+2)}$$

Thus

$$a_n \le \frac{9}{2}\cdot\frac{3}{n} \to 0 \quad \text{as} \quad n\to\infty.$$

So

$$\lim_{n\to\infty} a_n = 0 \quad \text{since} \quad a_n \ge 0.$$

QR
3. (a) $\sum_{n=0}^{\infty} x^n$ is geometric with $|r| = |x|$.

Thus the sum converges for $|x| < 1$, diverges for $|x| > 1$

That is,

$-1 < x < 1$ converges

$x > 1$ or $x < -1$ diverges

(a) $$\sum_{n=3}^{\infty} \frac{2e^n}{\pi^{n+1}} = \sum_{n=3}^{\infty} \frac{2}{\pi}\left(\frac{e}{\pi}\right)^n = \frac{2}{\pi}\left(\frac{e}{\pi}\right)^3 + \frac{2}{\pi}\left(\frac{e}{\pi}\right)^4 + \ldots$$

Geom. $a = \frac{2}{\pi}\left(\frac{e}{\pi}\right)^3 \quad r = \frac{e}{\pi}$

$|r| = \frac{e}{\pi} < 1$, so the series converges

to

$$\frac{\frac{2}{\pi}\left(\frac{e}{\pi}\right)^3}{1-\frac{e}{\pi}} = \frac{2e^3/\pi^4}{\frac{\pi-e}{\pi}} = \frac{2e^3}{(\pi-e)\pi^3}$$

(b) $\sum_{K=0}^{\infty} \tanh(K+2) - \tanh K$

$S_n = \sum_{K=0}^{n} \tanh(K+2) - \tanh K$

$= (\cancel{\tanh 2} - \overset{\nearrow 0}{\cancel{\tanh 0}}) + (\cancel{\tanh 3} - \tanh 1)$

$+ (\cancel{\tanh 4} - \cancel{\tanh 2}) + (\cancel{\tanh 5} - \cancel{\tanh 3})$

$+ (\cancel{\tanh 6} - \cancel{\tanh 4}) + \_\_\_\_$

$(\cancel{\tanh n} - \cancel{\tanh(n-2)}) + (\tanh(n+1) - \cancel{\tanh(n-1)})$

$+ (\tanh(n+2) - \cancel{\tanh(n)})$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} (-\tanh 1 + \tanh(n+1) + \tan(n+2))$

$= 2 - \tanh 1$

4. (a) $\lim_{K\to\infty} \frac{1}{\pi} \cos\left(\frac{K}{K^2+1}\right)$

$= \frac{1}{\pi} \cos\left(\lim_{K\to\infty} \frac{K}{K^2+1}\right) = \frac{1}{\pi} \cos 0 = \frac{1}{\pi} \neq 0$

Thus the sum diverges by the test for divergence.

(b) $\sum_{n=1}^{\infty} \frac{n+1}{\sqrt{n^4+e}} > 0$ L. Comp. with $b_n = \frac{n}{n^2} = \frac{1}{n} > 0$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n+1}{\sqrt{n^4+e}}}{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{n(n+1)}{\sqrt{n^4+e}} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{\sqrt{1+\frac{e}{n^4}}}$$

$$= 1 > 0.$$

Thus $\sum_{n=1}^{\infty} \frac{n+1}{\sqrt{n^4+e}}$ diverges since $\sum \frac{1}{n}$ does.

(c) $\sum_{j=1}^{\infty} \frac{e^{2j}\cos j}{j!}$

Ratio Test.. $\cos j$ may be negative.
Prove absolutely convergent first.

$$0 \le \left| \frac{e^{2j}\cos j}{j!} \right| \le \frac{e^{2j}}{j!} \quad \forall j \in \mathbb{N}$$

$\sum \frac{e^{2j}}{j!}$ converges by Ratio Test:

$$\lim_{j\to\infty} \left| \frac{a_{j+1}}{a_j} \right| = \lim_{j\to\infty} \frac{e^{2j+2}}{(j+1)!} \frac{j!}{e^{2j}}$$

$$= \lim_{j\to\infty} \frac{e^2 j!}{(j+1)!} = \lim_{j\to\infty} \frac{e^2}{j+1} = 0 < 1$$

Thus $\sum_{j=1}^{\infty} \left| \frac{e^{2j}\cos j}{j!} \right|$ converges by comparison

$\therefore \sum_{j=1}^{\infty} \frac{e^{2j}\cos j}{j!}$ is

absolutely convergent and thus it is convergent.

5. $f(x) = \frac{1}{x}$ is positive, cont. on $[1, \infty)$

It's decreasing since $\frac{1}{x+1} < \frac{1}{x}$ for $x \geq 1$.

$$\int_1^{\infty} \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx$$

$$= \lim_{t\to\infty} [\ln x]_1^t = \lim_{t\to\infty} (\ln t - \ln 1)$$

$= \infty$, so diverges.

Hence the corresponding $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

6. $\int 2\pi x\, ds = SA$

$$\frac{dy}{dx} = \frac{x^3}{4} - \frac{1}{x^3}$$

$$= \int 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{x^6}{16} - \frac{2}{4} + \frac{1}{x^6}$$

$$= \int_1^2 2\pi x \sqrt{\frac{1}{2} + \frac{x^6}{16} + \frac{1}{x^6}}\, dx$$

$$= \frac{x^6}{16} + \frac{1}{x^6} - \frac{1}{2}$$

$$= \int_1^2 2\pi x \sqrt{\left(\frac{x^3}{4} + \frac{1}{x^3}\right)^2}\, dx$$

$$= \int_1^2 2\pi x \left(\frac{x^3}{4} + \frac{1}{x^3}\right) dx$$

$$= 2\pi \int_1^2 \left(\frac{x^4}{4} + \frac{1}{x^2}\right) dx$$

$$= 2\pi \left[\frac{x^5}{20} - \frac{1}{x}\right]_1^2$$

$$= 2\pi \left[\frac{2^5}{20} - \frac{1}{2} - \left(\frac{1}{20} - 1\right)\right]$$

$$= 2\pi \left[\frac{32}{20} - \frac{1}{20} + \frac{1}{2}\right] = 2\pi \left(\frac{41}{20}\right) = \frac{41\pi}{10}$$