Quiz I.

1. e is the unique real number st $\ln e = 1$.

2. (i) $f'(x) = 2 + (f(x))^2 > 0$. Hence f is increasing, so f is 1-1. Since f is 1-1, it has an inverse function.

(ii) $$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))} = \frac{1}{2 + [f(f^{-1}(y))]^2}$$

$$= \frac{1}{2+y^2}$$

3. (i) $$f'(x) = \tfrac{2}{3}(x\ln x)^{-1/3}\left(\frac{x}{x} + \ln x\right)$$
$$= \tfrac{2}{3}(x\ln x)^{-1/3}(1+\ln x)$$

(ii) $$g'(x) = \frac{(\ln x)\sec^2(\ln x)\frac{1}{x} - \tan(\ln x)\frac{1}{x}}{(\ln x)^2}$$

4. $$\int_0^e \frac{x^3+3x^2}{x^4+4x^3+e^3}\,dx$$

Let $u = x^4 + 4x^3 + e^3$

$du = (4x^3 + 12x^2)\,dx$

$du = 4(x^3+3x^2)\,dx$

$\frac{1}{4}du = (x^3+3x^2)\,dx$

when $x = e$, $u = e^4 + 5e^3$

$x = 0$, $u = e^3$

$$= \frac{1}{4}\int_{e^3}^{e^4+5e^3} \frac{1}{u}\,du$$

$$= \frac{1}{4}\left[\ln|u|\right]_{e^3}^{e^4+5e^3}$$

$$= \frac{1}{4}\left[\ln(e^4+5e^3) - \ln e^3\right] = \frac{1}{4}\ln\left(\frac{e^4+5e^3}{e^3}\right)$$

$$= \frac{1}{4}\left[\ln(e^4+5e^3) - 3\right] \quad = \frac{1}{4}\ln(e+5)$$

Math 222 Fall 2010 $\boxed{\text{KEY}}$

Quiz II Name Amanda Taylor

1. For $a>0$, b any real number, define a^b.

1 $e^{b\ln a}$

2. Solve for x: $2^{x^2}=e^x$

$x^2\ln 2 = \ln e^x$ ½

2 $x^2 \ln 2 = x$ ½

$x^2\ln 2 - x = 0$

$x(x\ln 2 - 1) = 0$ ½

$x=0$ or $x\ln 2 - 1 = 0$

$x = \frac{1}{\ln 2}$ ½

−½ for missing solutions.

3. Calculate the indicated derivatives:

1.5 (i) $f(x) = x^{\sqrt{2}} - (\sqrt{2})^x$

$f'(x) = \sqrt{2}\, x^{(\sqrt{2}-1)} - \sqrt{2}^{\,x} \ln(\sqrt{2})$

½ 1

2 (ii) $g(x) = \arcsin(\cosh x + \cos x)$

$g'(x) = \dfrac{1}{\sqrt{1-(\cosh x + \cos x)^2}} \cdot (\sinh x - \sin x)$

1 1

4. Evaluate the integral:

$\int \left(\dfrac{\arccos x}{1-x^2}\right)^{1/2} dx$

2

Let $u = \arccos x$

$du = -\dfrac{1}{\sqrt{1-x^2}}\,dx$

$-du = \dfrac{1}{\sqrt{1-x^2}}\,dx$

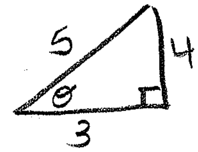
$= \int \dfrac{(\arccos x)^{\frac{1}{2}}}{\sqrt{1-x^2}}\,dx$

$= -\int u^{\frac{1}{2}}\,du = \frac{2}{-3}u^{\frac{3}{2}} + C = \dfrac{-2}{3}(\arccos x)^{\frac{3}{2}} + C$

1 1

5. Calculate $\csc(\arccos \frac{3}{5})$.

$\arccos\left(\frac{3}{5}\right) = \theta$ ½



$25 - 9 = 16$

½

1.5 $\cos\theta = \frac{3}{5}$

$\csc\theta = \dfrac{1}{\sin\theta} = \dfrac{1}{\frac{4}{5}} = \dfrac{5}{4}$ ½

Quiz 3 $\boxed{\text{Key}}$

1. $\int \ln x \, dx$

let $u = \ln x$ $\quad dv = dx$

$du = \frac{1}{x} dx$ $\quad v = x$

$= x \ln x - \int \frac{x}{x} dx$

$= x \ln x - \int dx = x \ln x - x + c$

2. $\int \cot^5\theta \sin^4\theta \, d\theta = \int \frac{\cos^5\theta}{\sin^5\theta} \sin^4\theta \, d\theta$

$= \int \frac{\cos^5\theta}{\sin\theta} d\theta = \int \frac{(1-\sin^2\theta)^2 \cos\theta}{\sin\theta} d\theta$

let $u = \sin\theta$

$du = \cos\theta \, d\theta$

$= \int \frac{(1-u^2)^2}{u} du = \int \frac{1-2u^2+u^4}{u} du = \int \left(\frac{1}{u} - 2u + u^3\right) du$

$= \ln|u| - u^2 + \frac{1}{4}u^4 + c = \ln|\sin\theta| - \sin^2\theta + \frac{1}{4}\sin^4\theta + c$

Quiz 4 key

1. $\int \sqrt{3-x-x^2}\,dx$

$= \int \sqrt{3-(x+x^2)}\,dx = \int \sqrt{3-(x+\frac{1}{2})^2+\frac{1}{4}}\,dx$

$= \int \sqrt{\frac{13}{4}-(x+\frac{1}{2})^2}\,dx$ let $x+\frac{1}{2} = \sqrt{\frac{13}{4}}\sin\theta$ where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$dx = \sqrt{\frac{13}{4}}\cos\theta\,d\theta$

+2 complete the sq

+1 trig subst.

+1 solving int.

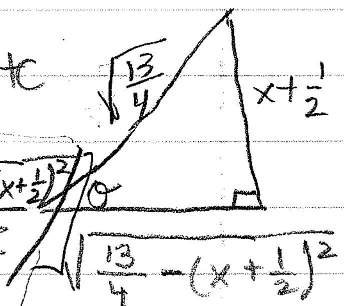
+1 putting in terms of x again

$= \int \frac{13}{4}\cos\theta\cos\theta\,d\theta$

$= \frac{13}{4}\int \cos^2\theta\,d\theta = \frac{13}{4}\int\left(\frac{1}{2}(1+\cos 2\theta)\right)d\theta$

$= \frac{13}{8}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$

$= \frac{13}{8}\left(\sin^{-1}\left(\frac{x+\frac{1}{2}}{\sqrt{\frac{13}{4}}}\right) + \sin\theta\cos\theta\right) + C$



$= \frac{13}{8}\left[\sin^{-1}\left(\frac{x+\frac{1}{2}}{\sqrt{\frac{13}{4}}}\right) + \left(\frac{x+\frac{1}{2}}{\sqrt{\frac{13}{4}}}\right)\left(\frac{\sqrt{\frac{13}{4}-(x+\frac{1}{2})^2}}{\sqrt{\frac{13}{4}}}\right)\right] + C$

2. $\int \frac{2x}{(x+3)(3x+1)}\,dx = \int\left(\frac{3/4}{x+3} - \frac{1/4}{3x+1}\right)dx$

$= \frac{3}{4}\ln|x+3| - \frac{1}{12}\ln|3x+1| + C$

+2 correct integration

+1 correct decomp.

$\underbrace{(x+3)(3x+1)}$ " $\frac{A}{x+3} + \frac{B}{3x+1}$

+2 correct constants

$2x = (3x+1)A + (x+3)B$

$x = -3 \quad -6 = -8A \quad A = \frac{3}{4}$

$x = -\frac{1}{3} \quad -\frac{2}{3} = \frac{8}{3}B \quad B = -\frac{1}{4}$

Quiz 5

Evaluate the integral or show it's divergent

1. $\int_1^{\infty} \frac{dx}{x \ln x} = \lim_{t\to\infty} \int_1^t \frac{dx}{x \ln x} = \lim_{t\to\infty} \lim_{s\to 1^+} \int_s^t \frac{dx}{x \ln x}$

$\lim_{t\to\infty} \lim_{s\to 1^+} \int_{x=s}^{x=t} \frac{1}{u}\, du$ Let $u = \ln x$, $du = \frac{1}{x} dx$

$= \lim_{t\to\infty} \lim_{s\to 1^+} \left[\ln|u|\right]_{x=s}^{x=t}$

$= \lim_{t\to\infty} \lim_{s\to 1^+} \left[\ln \ln x\right]_s^t = \lim_{t\to\infty} \lim_{s\to 1^+} (\ln\ln t - \ln\ln s)$

$\lim_{s\to 1^+} \ln\ln s = \lim_{u\to 0^+} \ln u = -\infty$, so the integral is divergent

2. Find the length of the curve

$x = \frac{1}{3}\sqrt{y}\,(y-3) \qquad 1 \le y \le 9$

$\frac{dx}{dy} = \frac{1}{2} y^{\frac{1}{2}} - \frac{1}{2} y^{-1/2}$

$\int_1^9 \sqrt{1 + \left(\frac{1}{2}\sqrt{y} - \frac{1}{2\sqrt{y}}\right)^2}\, dy = \int_1^9 \sqrt{1 + \frac{1}{4}y - \frac{1}{2} + \frac{1}{4y}}\, dy$

$= \int_1^9 \sqrt{\frac{1}{4}y + \frac{1}{2} + \frac{1}{4y}}\, dy = \int_1^9 \sqrt{\left(\frac{1}{2\sqrt{y}} + \frac{1}{2}\sqrt{y}\right)^2}$

$= \int_1^9 \left(\frac{1}{2\sqrt{y}} + \frac{1}{2}\sqrt{y}\right) dy$, then solve as usual.

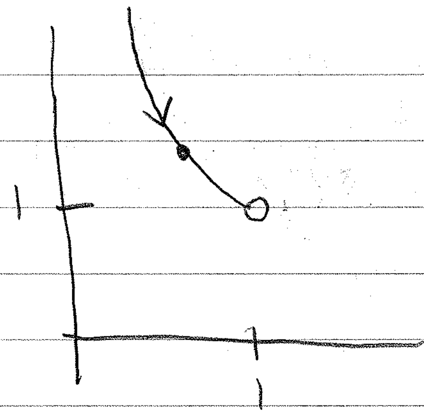
3. (a) eliminate the parameter to find a Cartesian eqn of the curve

(b) Sketch the curve and indicate the direction it is traced as the paramter increase. (w/ an arrow like in class)

$x = \sin t \quad y = \csc t \quad 0 < t < \pi/2$

(a) $y = \csc t = \frac{1}{\sin t} = \frac{1}{x}$ $\quad y = \frac{1}{x},\ y > 1$

(b)



t	x	y
$\to 0$	$\to 0$	$\to \infty$
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\sqrt{2}$
$\to \frac{\pi}{2}$	$\to 1$	$\to 1$

Test 1 Key

3 1. (a) $\ln x = \int_1^x \frac{1}{t}\,dt$

3 (b) e is the unique # st $\ln e = 1$

3 (c) $\sinh x = \dfrac{e^x - e^{-x}}{2}$

6 2. $e^{\pi^x} = 4^{e^x}$

$\pi^x \underbrace{\cancel{\ln e}}_{1} = e^x \ln 4$ 1

$x \ln \pi = \ln(e^x \ln 4)$ 1

$x \ln \pi = \ln e^x + \ln \ln 4$ 1

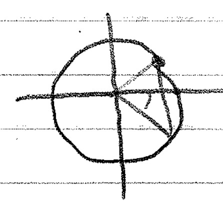
$x \ln \pi = x + \ln \ln 4$

$x \ln \pi - x = \ln \ln 4$ 1

$x(\ln \pi - 1) = \ln \ln 4$ 1

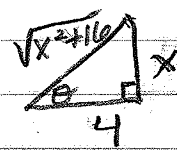
$x = \dfrac{\ln \ln 4}{\ln \pi - 1}$ 1

3 3. (a) $\cos^{-1}(\cos(-\frac{\pi}{4})) = \frac{\pi}{4}$



5 (b) $\sin(\underbrace{\tan^{-1}\frac{x}{4}}_{\theta})$

$\tan\theta = \frac{x}{4}$ 1



2

$\sin\theta = \dfrac{x}{\sqrt{x^2+16}}$ 2

3 (c) $\sinh(\ln 3) = \dfrac{e^{\ln 3} - e^{-\ln 3}}{2} = \dfrac{3 - \frac{1}{3}}{2} = \dfrac{\frac{8}{3}}{2} = \frac{4}{3}$

5 4. (a) $f'(x) = \dfrac{\ln x - x\cdot\frac{1}{x}}{(\ln x)^2} = \dfrac{\ln x - 1}{(\ln x)^2}$

5 (b) $g'(x) = \dfrac{1}{\tan^{-1}(x^2)}\cdot(\tan^{-1}x^2)'$

$= \dfrac{1}{\tan^{-1}(x^2)}\cdot\dfrac{1}{1+(x^2)^2}\cdot 2x$

$= \dfrac{2x}{\tan^{-1}(x^2)(1+x^4)}$

5 (c) $h'(x) = \cosh x + \sqrt{2}^{\,\sin x}(\ln\sqrt{2})\cos x$

5. 7 (a) $\displaystyle\int \frac{\cos(1-\ln x)}{x}\,du$ Let $u = 1-\ln x$

$du = -\frac{1}{x}\,dx$

$-du = \frac{1}{x}\,dx$

$= -\displaystyle\int \cos u\,du$

$= -\sin u + c = -\sin(1-\ln x) + c$

7 (b) $\displaystyle\int \sec^2 x\cdot 2^{\tan x}\,dx$ Let $u = \tan x$

$du = \sec^2 x\,dx$

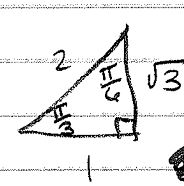
$= \displaystyle\int 2^u\,du = \frac{2^u}{\ln 2} + c = \frac{2^{\tan x}}{\ln 2} + c$

7 (c) $\displaystyle\int_0^{\ln\sqrt{3}} \frac{e^x}{1+e^{2x}}\,dx = \int_0^{\ln\sqrt{3}} \frac{e^x}{1+(e^x)^2}\,dx$ 1

$= \tan^{-1}(e^x)\Big]_0^{\ln\sqrt{3}}$ 3 $= \tan^{-1}\sqrt{3} - \tan^{-1} 1$

$= \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$

1 1 1



6 6. (a) $\lim_{x\to\infty} \frac{e^x}{x^2} \underset{\frac{\infty}{\infty}}{\overset{H}{=}} \lim_{x\to\infty} \frac{e^x}{2x} \underset{\frac{\infty}{\infty}}{\overset{H}{=}} \lim_{x\to\infty} \frac{e^x}{2} = \infty$

6 (b) $\lim_{x\to\infty} (x^2-1)e^{-x^2} = \lim_{x\to\infty} \frac{x^2-1}{e^{x^2}}$

$\underset{\frac{\infty}{\infty}}{\overset{H}{=}} \lim_{x\to\infty} \frac{2x}{2xe^{x^2}} = \lim_{x\to\infty} \frac{1}{e^{x^2}} = 0$

5 (c) $\lim_{x\to\infty} x\sin\frac{1}{x} \underset{0\cdot\infty}{=} \lim_{x\to\infty} \frac{\sin\frac{1}{x}}{1/x}$

$= \lim_{u\to 0} \frac{\sin u}{u} = 1$

$\boxed{\text{OR}}$

$\lim_{x\to\infty} x\sin\frac{1}{x} \underset{0\cdot\infty}{=} \lim_{x\to\infty} \frac{\sin\frac{1}{x}}{1/x} \underset{\frac{0}{0}}{\overset{H}{=}} \lim_{x\to\infty} \frac{\cancel{-\frac{1}{x^2}}\cos\frac{1}{x}}{\cancel{-1/x^2}}$

$= 1$

8 (d) $\lim_{x\to 0^+} (2x+1)^{\cot x} = \lim_{x\to 0^+} e^{\cot x \ln(2x+1)} = e^2$

$\lim_{x\to 0^+} \cot x \cdot \ln(2x+1) \underset{\infty\cdot 0}{=} \lim_{x\to 0^+} \frac{\ln(2x+1)}{\tan x}$

$\underset{\frac{0}{0}}{\overset{H}{=}} \lim_{x\to 0^+} \frac{\frac{2}{2x+1}}{\sec^2 x} = \lim_{x\to 0^+} \frac{2\cos^2 x}{2x+1} = \frac{2}{2\cdot 0+1} = 2$

3

2 1

7. (a) $f'(x) > 0$, so f is increasing,

5 hence 1-1. 1-1 functions have inverses.

1 1

(b) $(f^{-1})'(x) = \dfrac{1}{f'(f^{-1}(x))} = \dfrac{1}{e^x + x^4}$

5